On Testing the Necessary Conditions for Visibility Graphs of Simple Polygons

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Abstract

In this report we give a polynomial time algorithm to test the Necessary conditions 3 and 4 given by Ghosh[1] for visiblity graphs of simple polygons. We also show that to test the Necessary condition 4 it is enough to test the path symmetric and first path consistent conditions given by Abello and Kumar[2].

1 Introduction

An undirected graph $G(V, E)$ is called a *visibility graph* of a simple polygon P if the vertices $v_1, v_2, \ldots, v_n$ correspond to the vertices of the polygon P and the edge (v_i, v_j) occurs in E iff the corresponding vertices p_i and p_j are visible in P *i.e.* every point in the line joining the points lie inside the polygon or on the polygonal boundary. Four Necessary conditions for recognizing visibility graphs of simple polygons have been given by Ghosh [1]. An $O(n^2)$ is also given in the same paper to test for Necessary conditions 1 and 2.

Let the vertices of the visibility graph VG be numbered $1, 2, \ldots, n$ along the Hamiltonian cycle in the counter-clockwise order. A cycle $i_1, i_2, \ldots, i_k, i_1$ is said to be an *ordered cycle* if $i_1, i_2, \ldots, i_k$ preserve their order in the Hamiltonian cycle. A path $i_1, i_2, \ldots, i_k$ is said to be an *ordered path* if $i_1, i_2, \ldots, i_k$ preserve their order in the Hamiltonian cycle. Consider any 2 vertices i and j with $i < j$. If $\exists$ an edge (i, j) in VG then (i, j) is called a *visible pair*. Else, it is called an *invisible pair*. The set of vertices from i to j in the clockwise and anticlockwise direction are called the *upper* and *lower chain* of (i, j) respectively. A vertex a is said to be a *blocking vertex* for an invisible pair (i, j) if $\exists$ no visible pair (k, m) $\forall k \in chain(i, a-1)$ and $m \in chain(a+1, j)$.

If an invisible pair (i, j) has atmost one blocking vertex in each of it's upper and lower chains then it is called a *minimal invisible pair*. It has been shown in [1] that it is enough to consider minimal invisible pairs. Throughout this report whenever a pair of vetices (i, j) is considered, it is assumed that $i < j$ without loss of generality.

In this report we state the four Necessary conditions due to Ghosh[1].Two results have been given which can be used to test the *Necessary condition* 4 for a given assignment. We have given a polynomial time algorithm to test for the Necessary condition 4 based on the first result. Finally, we also suggest a method which could be used to construct a valid assignment.

2 Necessary Conditions

Four Necessary conditions have been established in [1] for visibility graphs of simple polygons. The first condition follows from the fact that every polygon has a triangulation.

Necessary Condition 1 *In a visibility graph every ordered cycle of $k \geq 4$ vertices has atleast $k-3$ diagonals.*

If a pair of vertices is invisible then their line of sight has to be blocked by atleast one vertex. This fact leads to the second condition.

Necessary Condition 2 *Every invisible pair (i, j) in a visibility graph has atleast one blocking vertex.*

An *assignment* is a mapping from minimal invisible pairs to vertices such that

1. A vertex assigned to any minimal invisible pair must be its blocking vertex.
2. Every minimal invisible pair is assigned to some blocking vertex and
3. If a is assigned to a minimal invisible pair (i, j) where $a \in$*lower chain*(i, j), then a is also assigned to all minimal invisible pairs (k, m) where $k \in$ *chain*$(i, a-1)$ and $m \in$ *chain*$(a+1, j)$.

Any vertex which is assigned to minimal invisible pair is a reflex vertex. Two invisible pairs $(i.j)$ and (k, l) are said to be *separable* with respect to a vertex a if k and l are encountered before i and j when the Hamiltonian cycle is traversed from a and a is a blocking vertex for both (i, j) and (k, l). In a polygon the same reflex vertex cannot block two separable invisible pairs simaltaneously. This suggests the third condition.

Necessary Condition 3 *There is an assignment of vertices to all minimal invisible pairs in a visibility graph such that no blocking vertex a is assigned to two or more minimal invisible pairs that are separable with respect to a.*

The sum of interior angles of an n-sided polygon is $(n-2) * 180$. So, in a polygon with n vertices the maximum number of reflex vertices possible is $n-3$. This suggests the fourth Necessary condition.

Necessary Condition 4 *For any assignment of blocking vertices to all minimal invisible pairs in a visibility graph, the total number of vertices of any ordered cycle C, which are assigned to the minimal invisible pairs between the vertices of C is atmost $|C|-3$.*

3 Algorithm to Test Necessary Conditions 3 and 4

Given an assignment if we are to test for the Necessary condition 4 by the brute force method (*i.e.*checking every ordered cycle) then it would take exponential time. Given a visibility graph VG and an assignment if the Necessary condition is violated for a particular ordered cycle C then this cycle has atmost 2 convex vertices (c_1 and c_2). So, C consists of the ordered path from c_1 to c_2 along the *lower* and the *upper chain*. Both these paths pass only through assigned blocking vertices. This leads to the following result that C is always the union of the shortest ordered path between c_1 and c_2 along the *lower* and the *upper chain*.

Theorem 1 *For a given visibility graph VG and an assignment if Necessary condition 4 is violated for some ordered cycle C ($|C| \geq 5$), then $\exists$ an invisible pair (i, j) between the vertices of C such that C is the union of the shortest ordered paths between i and j along the* lower *and the* upper chain.

Proof. Given the ordered cycle C in which the Necessary condition 4 is violated we know that there are atmost 2 convex vertices (say, c_1 and c_2) in C (Fig.1). Let the path from c_1 to c_2 in C along the *lower chain* be $P_1 : l_1, l_2, \ldots, l_k$. Consider any other ordered path $P_2 : v_1, v_2, \ldots, v_r$ from c_1 to c_2 along the *lower chain*.

Let $l_j = v_j$ for $j = 1, \ldots, i-1$ and $l_i \neq v_i$.

Case 1. $v_i > l_i$

l_j is an assigned blocking vertex for $j = 1, \ldots, k$. So,l_i blocks (l_{j-1}, l_{j+1}) where $l_0 = c_1$ and $l_{k+1} = c_2$. Now, l_i blocks (l_{i-1}, l_{i+1}) and l_{i+1} blocks (l_i, l_{i+2}) $\Rightarrow l_i$ blocks (l_{i-1}, l_{i+2}) by Lemma 2 of [1]. Using an analogous arguement we get l_i blocks (c_1, c_2).
$\Rightarrow \nexists$ any visible pair of the type (a, b) where $a \in (c_1, l_{i-1})$ and $b \in (l_{i+1}, c_2)$. So, $v_i > l_i$ is not possible because then (v_{i-1}, v_i) will be a visible pair of the above type (a, b).

Case 2. $v_i < l_i$

P_2 has to pass through l_i because due to the same reason as in the above case. So, P_2 has greater path length than P_1. Now, since P_2 is arbitrary this implies that P_1 is the shortest ordered path along the *lower chain*.

By the same arguement we can show that the ordered path along the *upper chain* in C is the shortest. **Q.E.D.**

An algorithm to test the fourth Necessary condition, given an assignment, follows straight from the above theorem. For every invisible pair consider the

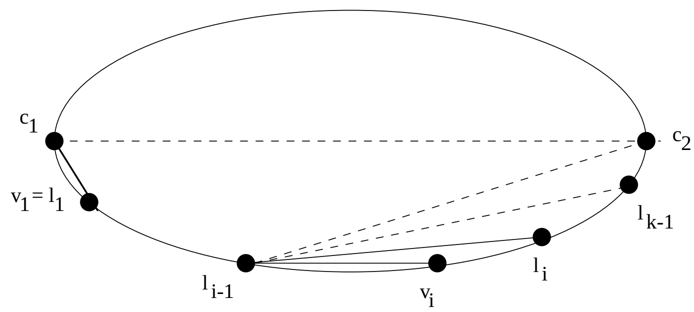

Figure 1: v_i cannot occur after l_i

shortest ordered path along the *lower* and *upper chain*. If any one of these paths are not unique then the Necessary condition 4 can't be violated for the ordered cycle so formed. In this case consider the next invisible pair. If the paths are unique, test the condition by explicitly counting the number of vertices that are assigned to minimal invisible pairs between the vertices of the ordered cycle. For a given invisible pair, finding the shortest path along either the *lower* or *upper chain* takes linear time on the number of vertices. The procedure described above has to be repeated for all non-minimal invisible pairs. Therefore, the algorithm takes $O(n^3)$ time to test the necessary condition 4 for a given assignment where n is the number of vertices in the visibility graph.

Abello and Kumar [2] have given four Necessary conditions for visibility graphs based on the Euclidean shortest path between vertices. It has been shown in [1] that all the Necessary conditions given in [2] follow from the necessary conditions 3 and 4. Here we show that two of the conditions given in [2] *i.e. path symmetric* condition and *first path consistent* condition together imply Necessary condition 4.

The *path symmetric* condition says that $ESP(i, j)$, the Euclidean shortest path from vertex i to j is same as $ESP(j, i)$. The *first path consistent* condition states that if $ESP(i, j)$ passes through k and $ESP(k, l)$ passes through j then $ESP(i, l)$ passes through k and j.

Theorem 2 *If the given graph VG satisfies first path consistent condition but violates Necessary condition* 4 *then the path symmetric condition is violated.*

Proof: Consider the ordered cycle C which has atmost 2 vertices which are not assigned to any minimal invisible pair between the vertices of C. Such an ordered cycle exists as Necessary condition 4 is violated.

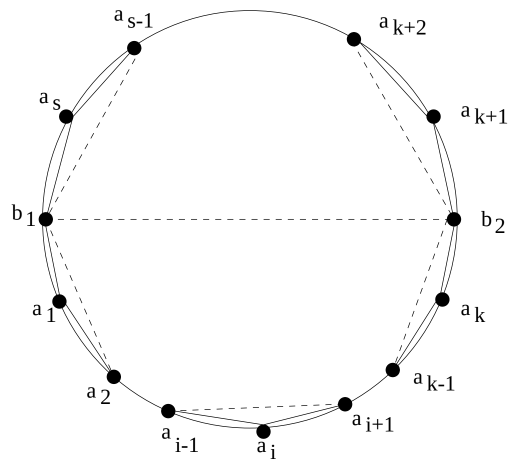

Figure 2: b_1 and b_2 are not adjacent

This ordered cycle might have $0, 1$ or 2 vertices which are not assigned to any minimal invisible pair within the cycle. Consider one vertex v which is assigned to a minimal invisible pair within the cycle. Now, $(v-1, v+1)$ should be an invisible pair and v should block it. This means that there are atleast $n-2$ invisible pairs of the type $(a, a+2)$ blocked by $a+1$. Morover, v lies in the Euclidean shortest path from $v-1$ to $v+1$. Infact, $ESP(v-1, v+1)$ is $v-1, v, v+1$.

Case 1. Number of vertices which are not assigned to any minimal invisible pair between the vertices of C, is 2 (say b_1 and b_2).
Case 1.1. b_1 and b_2 are not adjacent on the hamiltonian cycle.

Let the hamiltonian cycle be $b_1, a_1, a_2, \ldots, a_k, b_2, a_{k+1}, \ldots, a_s, b_1$ (Fig.2) where a_i blocks the invisible pair (a_{i-1}, a_{i+1}) for $i = 2, \ldots, k-1, k+2, \ldots, s-1$ and a_1 blocks (b_1, a_2); a_k blocks (a_{k-1}, b_2); a_{k+1} blocks (b_2, a_{k+2}) and a_s blocks (b_1, a_{s-1}).

Now, $ESP(b_1, a_2)$ is $b_1, a_1, a_2 \Rightarrow$ it passes through a_1. Also, $ESP(a_1, a_3)$ passes through a_2. Therefore, by the first path consistent condition $ESP(b_1, a_3)$ passes through a_1 and a_2. Using the same arguement repeatedly, $ESP(b_1, b_2)$ passes through $a_1, a_2, \ldots, a_k$. Now, $\exists$ a path from b_1 to b_2 passing only through $a_1, a_2, \ldots, a_k$ (along the Hamiltonian cycle). Therefore, $ESP(b_1, b_2)$ is $b_1, a_1, a_2, \ldots, a_k, b_2$. By a similar arguement we can show that $ESP(b_2, b_1)$ is $b_2, a_{k+1}, a_{k+2}, \ldots, a_s, b_1 \Rightarrow$ path symmetric condition is violated.

Case 1.2. b_1 and b_2 are adjacent to each other in the Hamiltonian cycle.

Let the Hamiltonian cycle be $b_2, b_1, a_1, \ldots, a_k, b_2$ similar to the above case (Fig.3). Consider $ESP(b_1, b_2)$. By the same arguement as above $ESP(b_1, b_2)$

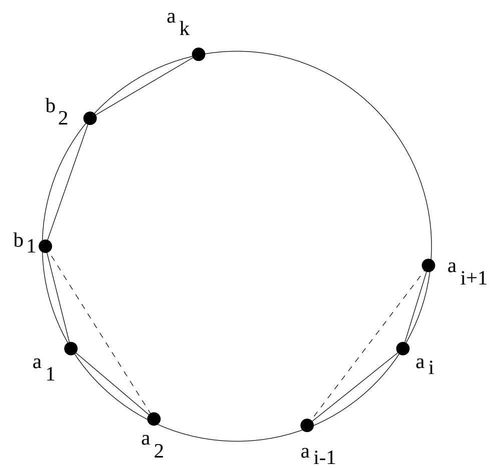

Figure 3: b_1 and b_2 are adjacent

is $b_1, a_1, a_2, \ldots, a_k, b_2$. But, (b_1, b_2) is a visible pair. Therefore, $ESP(b_2, b_1)$ is $b_2, b1 \Rightarrow$ path symmetric condition is violated.

Case 2. Number of vertices which are not assigned to any minimal invisible pair between the vertices of C is one (say b_1).

Let the hamiltonian cycle be $b_1, a_1, \ldots, a_k, b_1$ (Fig.4). Consider $ESP(b_1, a_k)$. By the same arguement as in *subcase B.* we can show that the path symmetric property is violated.

Case 3. All the vertices are assigned to some minimal invisible pair between the vertices of C.

Let the hamiltonian cycle be $a_1, \ldots, a_k, a_1$ (Fig.5). Consider $ESP(a_1, a_k)$. By the same arguement as in *case* 2. we can show that the path symmetric property is violated. **Q.E.D.**

The above theorem suggests that, to test the *Necessary condition* 4 it is enough to test both *path symmetric* condition and the *first path consistent* condition.

Given a visiblity graph and an assignment, to test for the Necessary condition 3, we proceed as follows. For every vertex v which is assigned to some minimal invisible pair, get the list of the assigned minimal invisible pairs. Renumber the vertices modulo v. Let this list be $i_1, i_2, \ldots, i_k$. Set $P(u_1, v_1)$ and $Q(u_2, v_2)$ to i_1. Consider the minimal invisible pairs (x, y) from the list one by one. If $y \leq u_1$ or $x \geq v_2$ then (x, y) & P or (x, y) & Q respectively, are separable with respect to v. So, reject the assignment. Else, if $x > u_1$ set p_1 to (x, y) and if $y < u_2$ set p_2 to (x, y). The updation and the check has to be done only once for each minimal invisible pair as each minimal invisible pair occurs on exactly one of the lists mentioned above.

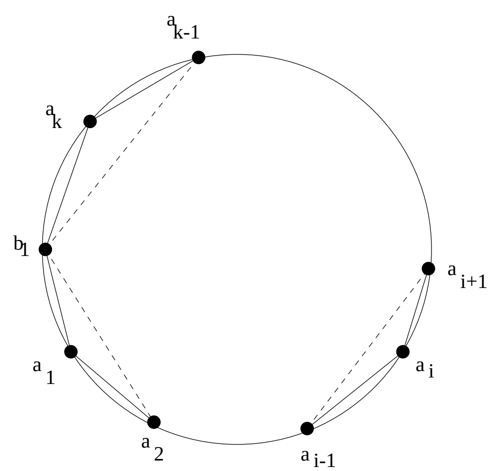

Figure 4: Only one vertex in C not assigned to any minimal invisible pair between the vertices of C

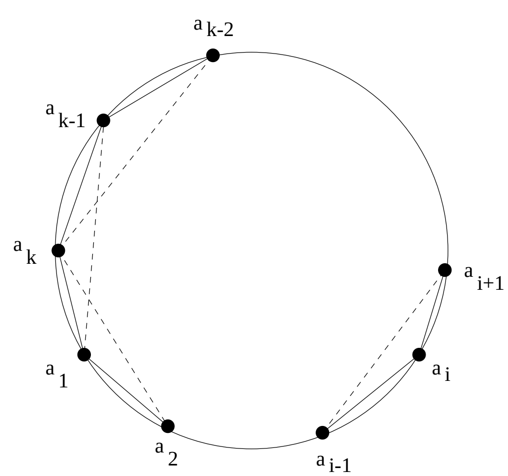

Figure 5: All vertices in C are assigned to some minimal invisible pair between the vertices of C

Therefore the test takes $O(n^2)$ time where n is the number of vertices in the visibility graph.

4 Conclusions

We have given polynomial time algorithms to test the Necessary conditions 3 and 4 for a visibility graph VG (of a simple polygon) for a given assignment. We are currently trying to generate the assignments which satisfy the Necessary conditions 3 and 4. A part of the assignment is always fixed as follows:

The minimal invisible pairs which have a sole blocking vertex have to be assignned only to that vertex. Whenever a minimal invisible pair (i, j) is assigned a vertex a we have to assign a to all the minimal invisible pairs (k, m) where $k \in (i, a-1)$ and $m \in (a+1, j)$. Also, a cannot be assigned to those minimal invisible pairs (x, y) where (x, y) and (i, j) are separable with respect to a. This makes all these minimal invisible pairs (x, y) have a single blocking vertex which can be assigned to them. The procedure has to be repeated iteratively till the remaining minimal invisible pairs have exactly 2 choices of vertices to be assigned to them. Say, (i, j) is one of the remaining minimal invisible pair which can be assigned to either k or l. It seems as though it is enough to check if there is a valid assignment where (i, j) is assigned to l. This is done by assigning l to as many minimal invisible pairs of the type (i, j') as possible. This remains to be proved.

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