A Statistical Convergence Perspective of Algorithms for Rank Aggregation from Pairwise Data

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Abstract

There has been much interest recently in the problem of rank aggregation from pairwise data. A natural question that arises is: under what sorts of statistical assumptions do various rank aggregation algorithms converge to an 'optimal' ranking? In this paper, we consider this question in a natural setting where pairwise comparisons are drawn randomly and independently from some underlying probability distribution. We first show that, under a 'time-reversibility' or Bradley-Terry-Luce (BTL) condition on the distribution, the rank centrality (PageRank) and least squares (HodgeRank) algorithms both converge to an optimal ranking. Next, we show that a matrix version of the Borda count algorithm, and more surprisingly, an algorithm which performs maximum likelihood estimation under a BTL assumption, both converge to an optimal ranking under a 'low-noise' condition that is strictly more general than BTL. Finally, we propose a new SVM-based algorithm for rank aggregation from pairwise data, and show that this converges to an optimal ranking under an even more general condition that we term 'generalized low-noise'. In all cases, we provide explicit sample complexity bounds for exact recovery of an optimal ranking. Our experiments confirm our theoretical findings and help to shed light on the statistical behavior of various rank aggregation algorithms.

1. Introduction

Rank aggregation is a classical problem that has been studied in several contexts, starting with social choice theory in 18th century France (Borda, 1781; Condorcet, 1785), and more recently, in computer science, statistics, linear algebra, and optimization, with a variety of different applications, and with different forms of both input rankings and desired aggregated rankings being considered (Dwork et al., 2001; Hochbaum, 2006; Meila et al., 2007; Ailon et al., 2008; Klementiev et al., 2008; Jagabathula & Shah, 2008; Guiver & Snelson, 2009; Ailon, 2010; Qin et al., 2010; Jiang et al., 2011; Gleich & Lim, 2011; Volkovs & Zemel, 2012; Negahban et al., 2012; Soufiani et al., 2012; Osting et al., 2013). A prominent setting that has gained interest in recent years is that of rank aggregation from *pairwise* data, where there is a set of n items to rank (such as movies or webpages), and one is given outcomes of various pairwise comparisons among these items (such as pairwise movie or webpage preferences of users); the goal is to aggregate these pairwise comparisons into a global ranking over the items. Various algorithms have been studied for this problem, including maximum likelihood under a Bradley-Terry-Luce (BTL) model assumption, rank centrality (PageRank/MC3) (Negahban et al., 2012; Dwork et al., 2001), least squares (HodgeRank) (Jiang et al., 2011), and a pairwise variant of Borda count (Borda, 1781; Jiang et al., 2011) among others.

In this paper, we consider statistical convergence properties of these rank aggregation algorithms under a natural statistical model, under which pairwise comparisons are drawn i.i.d. from some fixed but unknown probability distribution. An 'optimal' ranking is then one which minimizes the probability of disagreement with a random pairwise comparison drawn from this distribution. We consider three conditions of increasing generality on the distribution: a BTL condition, a 'low-noise' (LN) condition similar to a condition considered by (Duchi et al., 2010) in a different setting, and a 'generalized low-noise' (GLN) condition. We show that the rank centrality and least squares algorithms both converge (in probability) to an optimal ranking under the BTL condition, and that the Borda count and BTL-ML algorithms converge to an optimal ranking under the LN condition; we then propose a new SVM based rank aggregation algorithm which we show converges to an optimal ranking under the more general GLN condition. In all cases, we obtain explicit sample complexity bounds.

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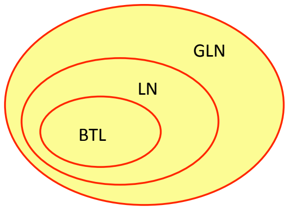

	BTL	LN	GLN
RANK CENTRALITY (NEGAHBAN ET AL., 2012)	√	×	×
LEAST SQUARES (HODGERANK) (JIANG ET AL., 2011)	√	×	×
BORDA COUNT (BORDA, 1781; JIANG ET AL., 2011)	√	√	×
BTL-ML (CLASSICAL)	√	√	×
SVM-RANKAGGREGATION (THIS PAPER)	√	√	√

Figure 1. We consider three increasingly general conditions on the distribution generating pairwise comparisons: BTL, LN, and GLN. The table summarizes our results on convergence of various rank aggregation algorithms to an optimal ranking under these conditions.

Related Work. The work most closely related to ours is that of Negahban et al. (Negahban et al., 2012), who analyzed convergence of the rank centrality algorithm under a statistical model where a fixed set of item pairs is repeatedly compared a fixed number of times, and the outcomes of the comparisons are determined by a BTL model. Our statistical model, where pairs to be compared are drawn randomly, is more natural in many applications (e.g. movie rankings). Our analysis of the rank centrality algorithm builds on that of (Negahban et al., 2012). However, we cannot use the standard matrix concentration tools used in (Negahban et al., 2012) since the comparison matrix in our case does not contain independent entries; instead, we use a McDiarmid-like concentration result of Kutin (Kutin, 2002) to analyze each entry separately. We point out our setting differs from the active learning settings of (Ailon, 2011; Jamieson & Nowak, 2011), where the goal is to recover a true permutation on n items by actively querying specific pairs; in our setting, the pairs are randomly sampled. Similarly, our setting differs from that of (Wauthier et al., 2013), where each pair of items can be compared at most once; in the rank aggregation setting we consider, it is common to have the same pair of items compared several times (with possibly different random outcomes). Our setting also differs from standard learning-to-rank problems involving pairwise preferences, where algorithms such as RankSVM or RankBoost are typically applied (Herbrich et al., 2000; Joachims, 2002; Freund et al., 2003), as there are no feature vectors in our setting; instead we simply have a finite number of objects with identifiers $1, \ldots, n$. Finally, our setting also differs from the subset ranking settings studied recently in machine learning and information retrieval (Cossock & Zhang, 2008; Duchi et al., 2010; Liu, 2011), where one ranks documents for various queries.

Summary and Organization. Figure 1 summarizes our results. Section 2 gives preliminaries. Section 3 summarizes various useful properties of the comparison matrix in our setting. Sections 4–6 consider conditions of increasing generality on the probability distribution generating pairwise comparisons, and analyze convergence properties of various rank aggregation algorithms under these conditions. Section 7 gives our experimental results. All proofs can be found in the supplementary material.

2. Preliminaries, Notation, and Background

Setup. Let $[n] = \{1, \ldots, n\}$ denote a set of n items to rank. Let $\mathcal{X} = \big\{(i,j) : i,j \in [n], i < j\big\}$. The learner is given a training sample $S = ((i_1, j_1, y_1), \ldots, (i_m, j_m, y_m)) \in (\mathcal{X} \times \{0,1\})^m$, where for each $k \in [m]$, $(i_k, j_k) \in \mathcal{X}$ denotes the k-th pair of items compared, and $y_k \in \{0,1\}$ denotes the outcome of the comparison; we adopt the convention that y_k is 1 if item j_k is ranked higher than item i_k, and 0 otherwise. Given S, the goal of the learner is to produce a ranking or *permutation* of the n items, $\sigma \in \mathcal{S}_n$.

We assume a probability distribution μ on $\mathcal{X}$ from which item pairs are sampled. For each $i < j$, we denote by μ_{ij} the probability of the pair (i,j) under μ; with some abuse of notation, we also denote $\mu_{ji} = \mu_{ij}\ \forall\ i < j$. We also assume a set of conditional label probabilities from which labels are drawn. Specifically, for each $i < j$, we denote by $P_{ij} \in [0,1]$ the probability that item j will be ranked higher than item i when items i and j are compared; we represent this as a pairwise preference matrix $\mathbf{P} \in [0,1]^{n \times n}$ with $P_{ji} = 1 - P_{ij}$ for $i < j$ and $P_{ii} = 0$. The training sample $S = ((i_1, j_1, y_1), \ldots, (i_m, j_m, y_m)) \in (\mathcal{X} \times \{0,1\})^m$ is then assumed to be drawn randomly according to $S \sim (\mu, \mathbf{P})^m$, i.e. the item pairs (i_k, j_k) are drawn randomly and independently according to μ, and conditioned on these, the labels are drawn as $y_k \sim \text{Bernoulli}(P_{i_k, j_k})$.

Given a distribution $(\mu, \mathbf{P})$ as above, define the expected pairwise disagreement error of a permutation $\sigma \in \mathcal{S}_n$ as

$$\mathrm{er}^{\mathrm{PD}}_{\mu,\mathbf{P}}[\sigma] \;=\; \sum_{i \neq j} \mu_{ij} P_{ij}\, \mathbf{1}\big(\sigma(i) < \sigma(j)\big)\,, \tag{1}$$

where $\mathbf{1}(\cdot)$ is 1 if its argument is true and 0 otherwise; this is the probability that σ does not agree with a pairwise comparison drawn randomly according to $(\mu, \mathbf{P})$. An 'optimal' permutation is then any permutation σ^* satisfying

$$\sigma^* \in \operatorname{argmin}_{\sigma \in \mathcal{S}_n} \mathrm{er}^{\mathrm{PD}}_{\mu,\mathbf{P}}[\sigma]\,. \tag{2}$$

Clearly, an ideal algorithm would recover (with high probability, for a large enough sample) such an optimal permutation. In what follows, we will consider various conditions on $(\mu, \mathbf{P})$, including a 'time-reversibility' or BTL condition, a 'low-noise' condition, and a 'generalized low-noise' condition, and will analyze convergence properties of various rank aggregation algorithms under these conditions.

All the algorithms we analyze take as input an *empirical pairwise comparison matrix* $\widehat{\mathbf{P}} \in [0,1]^{n \times n}$, which in our case is constructed from S as follows:

$$\widehat{P}_{ij} = \begin{cases} N_{ij}^{(1)}/N_{ij} & \text{if } i < j \text{ and } N_{ij} > 0 \\ 1 - \big(N_{ji}^{(1)}/N_{ji}\big) & \text{if } i > j \text{ and } N_{ji} > 0 \\ 0 & \text{otherwise;} \end{cases} \quad (3)$$

where

$$\begin{aligned} N_{ij} &= \textstyle\sum_{k=1}^m \mathbf{1}(i_k = i, j_k = j)\,; \\ N_{ij}^{(1)} &= \textstyle\sum_{k=1}^m \mathbf{1}(i_k = i, j_k = j, y_k = 1)\,. \end{aligned}$$

Note that the empirical comparison matrix $\widehat{\mathbf{P}}$ differs from the true pairwise preference matrix $\mathbf{P}$ in an important aspect: while $\mathbf{P}$ satisfies $P_{ij} + P_{ji} = 1$ for all $i < j$, in the case of $\widehat{\mathbf{P}}$, if a particular pair $i < j$ is not observed in S, we can have $\widehat{P}_{ij} = \widehat{P}_{ji} = 0$. To reinforce this distinction, we will use the terms pairwise *preference* matrix for $\mathbf{P}$ and pairwise *comparison* matrix for $\widehat{\mathbf{P}}$ throughout.

The structure of $\widehat{\mathbf{P}}$ is critical in our analysis: unlike (Negahban et al., 2012), where the empirical comparison matrix was constructed by fixing a priori some subset of pairs (i, j) to be compared and then repeatedly comparing each such pair a fixed number of times, which led to the entries of the matrix being independent, in our case, the entries of $\widehat{\mathbf{P}}$ are not independent (note that if some pair of items is sampled many times, other item pairs will be less frequent in S); therefore we cannot apply the matrix concentration tools used in (Negahban et al., 2012). Instead, we will show the elements of $\widehat{\mathbf{P}}$ satisfy a bounded differences property with high probability, allowing us to analyze $\widehat{\mathbf{P}}$ using Kutin's extension of McDiarmid's inequality (Kutin, 2002). This and other properties of $\widehat{\mathbf{P}}$, proved in Section 3, will then be used to analyze various algorithms in Sections 4–6.

Notation. We will find it convenient to define

$$\mu_{\min} \;=\; \min_{i<j} \mu_{ij}\,, \quad (4)$$

$$B(\mu_{\min}) \;=\; 3\big(\tfrac{12}{\mu_{\min}^2} + 3\big) \ln\big(\tfrac{12}{\mu_{\min}^2} + 3\big)\,. \quad (5)$$

Our results below will assume $\mu_{\min} > 0$. We will use capital boldface letters such as $\mathbf{P}$, $\mathbf{Q}$ for matrices and lower case boldface letters such as $\mathbf{f}$, $\boldsymbol{\pi}$ for vectors. For $\mathbf{f} \in \mathbb{R}^n$, we will denote by $\|\mathbf{f}\|_1 = \sum_{i=1}^n |f_i|$, $\|\mathbf{f}\|_2 = \big(\sum_{i=1}^n f_i^2\big)^{1/2}$, and $\|\mathbf{f}\|_\infty = \max_i |f_i|$ the standard L_1, L_2 and L_∞ norms. Also, for $\mathbf{f} \in \mathbb{R}^n$, we will denote by argsort$(\mathbf{f})$ the set of permutations that order items $i \in [n]$ in decreasing order of scores f_i, breaking ties arbitrarily:

$$\text{argsort}(\mathbf{f}) = \big\{\sigma \in \mathcal{S}_n : f_i > f_j \implies \sigma(i) < \sigma(j)\big\}\,.$$

Background Results. The following definition of strongly difference-bounded random variables and concentration result for such random variables, both due to Kutin (Kutin, 2002), will be used in our analysis of the empirical comparison matrix $\widehat{\mathbf{P}}$ in Section 3.

Definition 1 (Strong difference-boundedness (Kutin, 2002))**.** *Let $X = (X_1, \ldots, X_m)$ be a vector of independent random variables with X_i taking values in some set A_i, and let $A = A_1 \times \cdots \times A_m$. Let $\phi : A \to \mathbb{R}$ be any function. Let $b, c > 0$ and $\delta \in (0,1]$. The random variable $\phi(X)$ is said to be* strongly difference-bounded by (b, c, δ) *if $\exists B \subset A$ with $\mathbf{P}\big(X \in B\big) \leq \delta$ such that for each $k \in [m]$,*

$$\begin{aligned} \sup_{x \notin B, x'_k \in A_k} \big|\phi(x) - \phi(x_1, \ldots, x'_k, \ldots, x_m)\big| &\leq c \\ \sup_{x \in A, x'_k \in A_k} \big|\phi(x) - \phi(x_1, \ldots, x'_k, \ldots, x_m)\big| &\leq b\,. \end{aligned}$$

Theorem 2 ((Kutin, 2002))**.** *Let $X = (X_1, \ldots, X_m)$ be a vector of independent random variables with X_i taking values in some set A_i, and let $A = A_1 \times \cdots \times A_m$. Let $\phi : A \to \mathbb{R}$ be any function such that $\phi(X)$ is strongly difference-bounded by $\big(b, \frac{\lambda}{m}, \exp(-Km)\big)$. Let $0 < \epsilon \leq 2\lambda\sqrt{K}$. If $m \geq \max\big(\frac{b}{\lambda},\ 3\big(\frac{6}{K} + 3\big)\ln\big(\frac{6}{K} + 3\big)\big)$, then*

$$\mathbf{P}\left(\big|\phi(X) - \mathbf{E}\big[\phi(X)\big]\big| \geq \epsilon\right) \;\leq\; 4\exp(-m\epsilon^2/8\lambda^2)\,.$$

3. Properties of Comparison Matrix $\widehat{\mathrm{P}}$

The following lemma summarizes various useful properties of the empirical comparison matrix $\widehat{\mathbf{P}}$ that are used in our proofs. In particular, a key property is that for large enough m, the elements of $\widehat{\mathbf{P}}$ are strongly difference-bounded, allowing us to obtain concentration results for them.

Lemma 3. *Let $(\mu, \mathbf{P})$ be such that $\mu_{\min} > 0$. Let $S \sim (\mu, \mathbf{P})^m$, and let $\widehat{\mathbf{P}}$ be constructed from S as in Eq. (3).*

1. Let $i \neq j$. If $m \geq \frac{4}{\mu_{\min}}$, then $\widehat{P}_{ij}$ is strongly difference-bounded by $\Big(1, \frac{2}{m\mu_{\min}}, \exp\big(\frac{-m\mu_{\min}^2}{2}\big)\Big)$.

2. Let $i \neq j$. Let $0 < \epsilon < 2\sqrt{2}$. If $m \geq B(\mu_{\min})$, then

$$\mathbf{P}\Big(\big|\widehat{P}_{ij} - \mathbf{E}[\widehat{P}_{ij}]\big| \geq \epsilon\Big) \;\leq\; 4\exp\Big(\frac{-m\epsilon^2\mu_{\min}^2}{32}\Big)\,.$$

3. Let $i \neq j$. Let $\epsilon > 0$. If $m \geq \frac{1}{\mu_{\min}}\ln\big(\frac{1}{\epsilon}\big)$, then $\big|\mathbf{E}[\widehat{P}_{ij}] - P_{ij}\big| \leq \epsilon\,.$

4. Let $i \neq j$. Let $0 < \epsilon < 4\sqrt{2}$. If $m \geq \max\big(B(\mu_{\min}),\ \frac{1}{\mu_{\min}}\ln\big(\frac{2}{\epsilon}\big)\big)$, then

$$\mathbf{P}\Big(\big|\widehat{P}_{ij} - P_{ij}\big| \geq \epsilon\Big) \;\leq\; 4\exp\Big(\frac{-m\epsilon^2\mu_{\min}^2}{128}\Big)\,.$$

5. Let $P_{ij} \in (0,1)\ \forall i \neq j$, and let $P_{\min} = \min_{i \neq j} P_{ij}$. Let $\delta \in (0,1]$. If $m \geq \frac{1}{\mu_{\min}P_{\min}}\ln\big(\frac{n(n-1)}{\delta}\big)$, then with probability at least $1 - \delta$, $\widehat{P}_{ij} > 0\ \forall i \neq j$.

The proof of Part 1 makes use of Hoeffding's inequality and involves a somewhat detailed, careful case-by-case analysis. Part 2 then follows from Part 1 and Theorem 2. Part 3 follows by observing $\mathbf{E}[\widehat{P}_{ij}] = P_{ij}(1 - (1 - \mu_{ij})^m)$. Part 4 follows from Parts 2 and 3. Part 5 is straightforward. Details can be found in the supplementary material.

4. Time-Reversibility/BTL Condition

We first consider the following 'time-reversibility' and BTL conditions on the preference matrix $\mathbf{P}$:

Definition 4 (**Time-reversibility condition**). *We say the pairwise preference matrix* $\mathbf{P} \in [0,1]^{n\times n}$ *satisfies the* time-reversibility condition *if the Markov chain* $\mathbf{Q}$ *given by*

$$Q_{ij} = \begin{cases} \frac{1}{n} P_{ij} & \text{if } i \neq j \\ 1 - \frac{1}{n}\sum_{k\neq i} P_{ik} & \text{if } i = j \end{cases} \qquad (6)$$

is time-reversible, i.e. if $\mathbf{Q}$ *is irreducible and aperiodic and the stationary probability vector* $\boldsymbol{\pi}$ *of* $\mathbf{Q}$ *satisfies* $\pi_i Q_{ij} = \pi_j Q_{ji}\ \forall i,j \in [n]$.

Definition 5 (**Bradley-Terry-Luce (BTL) condition**). *We say the pairwise preference matrix* $\mathbf{P} \in [0,1]^{n\times n}$ *satisfies the* Bradley-Terry-Luce (BTL) condition *if it corresponds to a BTL model, i.e. if* $\exists \mathbf{w} \in \mathbb{R}^n_+$ *with* $w_i > 0\ \forall i$ *such that* $P_{ij} = w_j/(w_i + w_j)\ \forall i \neq j$.

Clearly, if $\mathbf{P}$ satisfies the time-reversibility condition with $\mathbf{Q}$ and $\boldsymbol{\pi}$ as above, then $\forall i \neq j$, $P_{ij} > P_{ji} \implies \pi_j > \pi_i$, and therefore any permutation that ranks items $i \in [n]$ in decreasing order of scores π_i is an optimal permutation w.r.t. the pairwise disagreement error (see Eq. (1)). Similarly, if $\mathbf{P}$ corresponds to a BTL model with parameter vector $\mathbf{w}$ as above, then any permutation that ranks items according to decreasing order of scores w_i is an optimal permutation. The following lemma shows that the time-reversibility and BTL conditions are in fact equivalent:

Lemma 6. *A preference matrix* $\mathbf{P} \in [0,1]^{n\times n}$ *satisfies the time-reversibility condition if and only if it satisfies the BTL condition.*

Note that if $\mathbf{P}$ satisfies the time-reversibility condition, then by the above result, $P_{ij} \in (0,1)\ \forall i \neq j$.

4.1. Convergence of Rank Centrality Algorithm

We start by analyzing convergence behavior of the rank centrality algorithm (Dwork et al., 2001; Negahban et al., 2012) (Algorithm 1)[1] in our setting under the above time-reversibility/BTL condition. In particular, we first show the following result, which establishes convergence of the score vector $\widehat{\boldsymbol{\pi}}$ produced by the rank centrality algorithm to $\boldsymbol{\pi}$, the stationary vector of the matrix $\mathbf{Q}$ defined in Eq. (6) (in L_2 norm):

[1]Note that the rank centrality algorithm as presented here differs slightly from (Negahban et al., 2012) in two aspects: rather than divide elements of $\widehat{\mathbf{P}}$ by the maximum degree, which in our case depends on the sample S, we divide by n in constructing $\widehat{\mathbf{Q}}$; similarly, since in our case the graph defining the Markov chain $\widehat{\mathbf{Q}}$ depends on S and may not be strongly connected, we allow for the possibility of producing a default vector $\widehat{\boldsymbol{\pi}} = \mathbf{0}$ in this case. Also, while (Negahban et al., 2012) are interested in the score vector $\widehat{\boldsymbol{\pi}}$, we are interested in the ordering $\widehat{\sigma}$ produced by $\widehat{\boldsymbol{\pi}}$.

Algorithm 1 Rank Centrality (PageRank/MC3) (Negahban et al., 2012; Dwork et al., 2001)

Input: Empirical comparison matrix $\widehat{\mathbf{P}} \in [0,1]^{n\times n}$

Construct an empirical Markov chain with transition probability matrix $\widehat{\mathbf{Q}}$ as follows:

$$\widehat{Q}_{ij} = \begin{cases} \frac{1}{n}\widehat{P}_{ij} & \text{if } i \neq j \\ 1 - \frac{1}{n}\sum_{k\neq i}\widehat{P}_{ik} & \text{if } i = j. \end{cases}$$

If $\widehat{\mathbf{Q}}$ defines an irreducible, aperiodic Markov chain, then compute $\widehat{\boldsymbol{\pi}}$, the stationary probability vector of $\widehat{\mathbf{Q}}$

else let $\widehat{\boldsymbol{\pi}} = \mathbf{0} \in \mathbb{R}^n_+$

Output: Permutation $\widehat{\sigma} \in \text{argsort}(\widehat{\boldsymbol{\pi}})$

Theorem 7. *Let* $(\mu, \mathbf{P})$ *be such that* $\mu_{\min} > 0$ *and* $\mathbf{P}$ *satisfies the BTL condition. Let* $\mathbf{Q}$ *be defined as in Eq. (6), and let* $\boldsymbol{\pi}$ *be the stationary probability vector of* $\mathbf{Q}$. *Let* $P_{\min} = \min_{i\neq j} P_{ij}$, $\pi_{\max} = \max_i \pi_i$, *and* $\pi_{\min} = \min_i \pi_i$. *Let* $0 < \epsilon \leq 1$ *and* $\delta \in (0,1]$. *If*

$$m \geq \max\left(\frac{1024\, n}{\epsilon^2 P_{\min}^2 \mu_{\min}^2}\left(\frac{\pi_{\max}}{\pi_{\min}}\right)^3 \ln\left(\frac{16n^2}{\delta}\right),\ B(\mu_{\min})\right),$$

then with probability at least $1-\delta$ *(over the random draw of* $S \sim (\mu, \mathbf{P})^m$ *from which* $\widehat{\mathbf{P}}$ *is constructed), the score vector* $\widehat{\boldsymbol{\pi}}$ *produced by the rank centrality algorithm satisfies*

$$\|\widehat{\boldsymbol{\pi}} - \boldsymbol{\pi}\|_2 \leq \epsilon\,.$$

The proof of Theorem 7 builds on the technique used by (Negahban et al., 2012), and makes use of two lemmas, which establish convergence of the empirical Markov chain $\widehat{\mathbf{Q}}$ to the true chain $\mathbf{Q}$ in spectral norm, and a lower bound on the spectral gap of $\mathbf{Q}$. As noted previously, the elements of $\widehat{\mathbf{P}}$, and therefore $\widehat{\mathbf{Q}}$, are not independent in our setting, and therefore we cannot apply the standard matrix concentration tools used in (Negahban et al., 2012). Our proof makes use of the strong difference-boundedness and related properties of $\widehat{\mathbf{P}}$ from Lemma 3 (details can be found in the supplementary material). From Theorem 7, we immediately have the following sample complexity bound for the rank centrality algorithm to exactly recover an optimal permutation under the time-reversibility/BTL condition:

Corollary 8. *Let* $(\mu, \mathbf{P})$ *be such that* $\mu_{\min} > 0$ *and* $\mathbf{P}$ *satisfies the BTL condition, and* $\exists (i \neq j) : P_{ij} \neq \frac{1}{2}$. *Let* $\mathbf{Q}$ *be defined as in Eq. (6), and let* $\boldsymbol{\pi}$ *be the stationary probability vector of* $\mathbf{Q}$. *Let* $P_{\min}$, $\pi_{\min}$ *and* $\pi_{\max}$ *be defined as in Theorem 7, and let* $r_{\min} = \min_{i,j:\pi_i\neq\pi_j} |\pi_i - \pi_j|$. *Let* $\delta \in (0,1]$. *If*

$$m \geq \max\left(\frac{9216\, n}{r_{\min}^2 P_{\min}^2 \mu_{\min}^2}\left(\frac{\pi_{\max}}{\pi_{\min}}\right)^3 \ln\left(\frac{16n^2}{\delta}\right),\ B(\mu_{\min})\right),$$

then with probability at least $1-\delta$ *(over the random draw of* $S \sim (\mu, \mathbf{P})^m$ *from which* $\widehat{\mathbf{P}}$ *is constructed), the permutation* $\widehat{\sigma}$ *output by the rank centrality algorithm satisfies*

$$\widehat{\sigma} \in \text{argmin}_{\sigma\in\mathcal{S}_n} \text{er}^{\text{PD}}_{\mu,\mathbf{P}}[\sigma]\,.$$

4.2. Convergence of Least Squares Algorithm

Next, we analyze convergence of the least squares (HodgeRank) algorithm (Jiang et al., 2011) (Algorithm 2) in our framework, again under the above time-reversibility/BTL condition on the preference matrix $\mathbf{P}$. As discussed in (Jiang et al., 2011), the pairwise comparison matrix $\widehat{\mathbf{P}}$ is converted to a skew-symmetric matrix $\widehat{\mathbf{Y}}$ via a log-odds ratio transform before applying least squares (see Algorithm 2). Similarly, given a pairwise preference matrix $\mathbf{P} \in [0,1]^{n\times n}$, we can define a skew-symmetric matrix $\mathbf{Y} \in \mathbb{R}^{n\times n}$ as

$$Y_{ij} = \begin{cases} \ln\left(\frac{P_{ij}}{P_{ji}}\right) & \text{if } i \neq j \text{ and } P_{ij} \in (0,1) \\ 0 & \text{otherwise.} \end{cases} \quad (7)$$

Let $E = \{(i,j) \in \mathcal{X} : P_{ij} \neq 0 \text{ or } P_{ji} \neq 0\}$, and let $\mathbf{f}^* \in \arg\min_{\mathbf{f}\in\mathbb{R}^n} \sum_{(i,j)\in E} \left((f_i - f_j) - Y_{ij}\right)^2$. Clearly, since $P_{ij} = 1 - P_{ji}\ \forall i \neq j$, we have $E = \mathcal{X}$. In this case, as discussed in (Jiang et al., 2011), the (minimum norm) solution to the above optimization problem is given by

$$f_i^* \;=\; -\frac{1}{n}\sum_{k=1}^{n} Y_{ik}\,. \quad (8)$$

The following lemma shows that if $\mathbf{P}$ satisfies the time-reversibility/BTL condition, then ranking items according to decreasing order of scores f_i^* as above yields an optimal ranking w.r.t. the pairwise disagreement error:

Lemma 9. *Let $(\mu, \mathbf{P})$ be such that $\mathbf{P}$ satisfies the BTL condition. Let $\mathbf{f}^* \in \mathbb{R}^n$ be defined as in Eq. (8). Then* $\text{argsort}(\mathbf{f}^*) \subseteq \text{argmin}_{\sigma\in\mathcal{S}_n} \text{er}^{\text{PD}}_{\mu,\mathbf{P}}[\sigma]$.

The following is our main result regarding convergence of the least squares algorithm. Note that it establishes convergence of the score vector $\widehat{\mathbf{f}}$ produced by the least squares algorithm to $\mathbf{f}^*$ (in L_∞ norm) under any $\mathbf{P}$ satisfying $P_{ij} \in (0,1)\ \forall i \neq j$; however the optimality of permutations obtained from $\mathbf{f}^*$ w.r.t. pairwise disagreement is guaranteed only when $\mathbf{P}$ satisfies the time-reversibility/BTL condition.

Theorem 10. *Let $(\mu, \mathbf{P})$ be such that $\mu_{\min} > 0$ and $P_{ij} \in (0,1)\ \forall i \neq j$. Let $\mathbf{Y} \in \mathbb{R}^{n\times n}$ and $\mathbf{f}^* \in \mathbb{R}^n$ be defined as in Eqs. (7) and (8). Let $P_{\min} = \min_{i\neq j} P_{ij}$. Let $0 < \epsilon \leq 1$ and $\delta \in (0,1]$. If*

$$m \geq \max\left(\frac{128}{P_{\min}^2 \mu_{\min}^2}\left(1 + \frac{2}{\epsilon}\right)^2 \ln\left(\frac{16n^2}{\delta}\right),\ B(\mu_{\min})\right),$$

then with probability at least $1-\delta$ (over the random draw of $S \sim (\mu,\mathbf{P})^m$ from which $\widehat{\mathbf{P}}$ is constructed), the score vector $\widehat{\mathbf{f}}$ produced by the least squares algorithm satisfies

$$\|\widehat{\mathbf{f}} - \mathbf{f}^*\|_\infty \leq \epsilon\,.$$

This immediately yields the following sample complexity bound for the least squares algorithm to exactly recover an optimal permutation under the BTL condition:

Algorithm 2 Least Squares/HodgeRank (Jiang et al., 2011)

Input: Empirical comparison matrix $\widehat{\mathbf{P}} \in [0,1]^{n\times n}$

Construct empirical skew-symmetric matrix $\widehat{\mathbf{Y}}$:

$$\widehat{Y}_{ij} = \begin{cases} \ln\left(\frac{\widehat{P}_{ij}}{\widehat{P}_{ji}}\right) & \text{if } i \neq j \text{ and } \widehat{P}_{ij} \in (0,1) \\ 0 & \text{otherwise.} \end{cases}$$

Let $\widehat{E} = \{(i,j) \in \mathcal{X} : \widehat{P}_{ij} \neq 0 \text{ or } \widehat{P}_{ji} \neq 0\}$

Compute $\widehat{\mathbf{f}} \in \text{argmin}_{\mathbf{f}\in\mathbb{R}^n} \sum_{(i,j)\in\widehat{E}} \left((f_j - f_i) - \widehat{Y}_{ij}\right)^2$

Output: Permutation $\widehat{\sigma} \in \text{argsort}(\widehat{\mathbf{f}})$

Corollary 11. *Let $(\mu, \mathbf{P})$ be such that $\mu_{\min} > 0$ and $\mathbf{P}$ satisfies the BTL condition, and $\exists (i \neq j) : P_{ij} \neq \frac{1}{2}$. Let $\mathbf{f}^*$ be as in Eq. (8), and let $r_{\min} = \min_{i,j: f_i^* \neq f_j^*} |f_i^* - f_j^*|$. Let $\delta \in (0,1]$. If*

$$m \geq \max\left(\frac{128}{P_{\min}^2 \mu_{\min}^2}\left(1 + \frac{6}{r_{\min}}\right)^2 \ln\left(\frac{16n^2}{\delta}\right),\ B(\mu_{\min})\right),$$

then with probability at least $1-\delta$ (over the random draw of $S \sim (\mu,\mathbf{P})^m$ from which $\widehat{\mathbf{P}}$ is constructed), the permutation $\widehat{\sigma}$ output by the least squares algorithm satisfies

$$\widehat{\sigma} \in \text{argmin}_{\sigma\in\mathcal{S}_n} \text{er}^{\text{PD}}_{\mu,\mathbf{P}}[\sigma]\,.$$

5. Low-Noise (LN) Condition

In this section we consider the following 'low-noise' condition on the preference matrix $\mathbf{P}$, which is similar to the condition studied by (Duchi et al., 2010) in a somewhat different context; as the lemma below shows, the low-noise condition includes the BTL condition as a special case.

Definition 12 (Low-noise (LN) condition). *We say the pairwise preference matrix $\mathbf{P} \in [0,1]^{n\times n}$ satisfies the* low-noise (LN) condition *if*

$$\forall i \neq j: \quad P_{ij} > P_{ji} \implies \sum_{k=1}^{n} P_{kj} > \sum_{k=1}^{n} P_{ki}\,.$$

Lemma 13. *If $\mathbf{P} \in [0,1]^{n\times n}$ satisfies the BTL condition, then it also satisfies the LN condition.*

For the rest of this section (Section 5), given a pairwise preference matrix $\mathbf{P} \in [0,1]^{n\times n}$, define $\mathbf{f}^* \in \mathbb{R}_+^n$ as

$$f_i^* \;=\; \frac{1}{n}\sum_{k=1}^{n} P_{ki}\,. \quad (9)$$

Clearly, if $\mathbf{P}$ satisfies the LN condition, then any permutation that ranks items $i \in [n]$ in descending order of scores f_i^* as defined above is an optimal permutation w.r.t. the pairwise disagreement error (see Eq. (1)).

5.1. Convergence of Borda Count Algorithm

Given a pairwise comparison matrix $\widehat{\mathbf{P}}$, (the matrix version of) the Borda count algorithm (Borda, 1781; Jiang et al., 2011) (Algorithm 3) simply averages for each item i the

Algorithm 3 Borda Count (Borda, 1781; Jiang et al., 2011)

Input: Empirical comparison matrix $\widehat{\mathbf{P}} \in [0,1]^{n \times n}$
For $i = 1$ to n: $\widehat{f}_i = \frac{1}{n}\sum_{k=1}^n \widehat{P}_{ki}$
Output: Permutation $\widehat{\sigma} \in \text{argsort}(\widehat{\mathbf{f}})$

fraction of times $\widehat{P}_{ki}$ it has beat each other item k, and ranks items by this score.[2] Here we show this algorithm converges to an optimal ranking under the LN condition.

The following result establishes convergence of the score vector $\widehat{\mathbf{f}}$ produced by Borda count to $\mathbf{f}^*$ (in L_∞ norm) under general $\mathbf{P}$; however the optimality of permutations obtained from $\mathbf{f}^*$ w.r.t. the pairwise disagreement error is guaranteed only for $\mathbf{P}$ satisfying the LN condition.

Theorem 14. *Let $(\mu, \mathbf{P})$ be such that $\mu_{\min} > 0$, and let $\mathbf{f}^* \in \mathbb{R}^n_+$ be defined as in Eq. (10). Let $0 < \epsilon \leq (4\sqrt{2})$ and $\delta \in (0,1]$. If*

$$m \geq \max\left(\frac{128}{\epsilon^2 \mu_{\min}^2} \ln\left(\frac{4n^2}{\delta}\right),\ B(\mu_{\min})\right),$$

then with probability at least $1-\delta$ (over the random draw of $S \sim (\mu, \mathbf{P})^m$ from which $\widehat{\mathbf{P}}$ is constructed), the score vector $\widehat{\mathbf{f}}$ produced by the Borda count algorithm satisfies

$$\|\widehat{\mathbf{f}} - \mathbf{f}^*\|_\infty \leq \epsilon\,.$$

This immediately yields the following sample complexity bound for the Borda count algorithm to exactly recover an optimal permutation under the LN condition:

Corollary 15. *Let $(\mu, \mathbf{P})$ be such that $\mu_{\min} > 0$ and $\mathbf{P}$ satisfies the LN condition, and $\exists (i \neq j) : P_{ij} \neq \frac{1}{2}$. Let $\mathbf{f}^*$ be as in Eq. (10), and let $r_{\min} = \min_{i,j: f_i^* \neq f_j^*} |f_i^* - f_j^*|$. Let $\delta \in (0,1]$. If*

$$m \geq \max\left(\frac{1152}{r_{\min}^2 \mu_{\min}^2} \ln\left(\frac{4n^2}{\delta}\right),\ B(\mu_{\min})\right),$$

then with probability at least $1-\delta$ (over the random draw of $S \sim (\mu, \mathbf{P})^m$ from which $\widehat{\mathbf{P}}$ is constructed), the permutation $\widehat{\sigma}$ output by the Borda count algorithm satisfies

$$\widehat{\sigma} \in \text{argmin}_{\sigma \in \mathcal{S}_n} \text{er}^{\text{PD}}_{\mu,\mathbf{P}}[\sigma]\,.$$

5.2. Convergence of BTL-ML Estimator

Given a pairwise comparison matrix $\widehat{\mathbf{P}}$, the BTL-ML estimator (Algorithm 4) finds a maximum likelihood score vector assuming a BTL model. Here we show this algorithm actually converges to an optimal permutation w.r.t. pairwise disagreement under the more general LN condition; in fact we obtain the same sample complexity bound for BTL-ML as for the Borda count algorithm above:

[2] The standard Borda count algorithm ranks items by the *number* of times they beat other items; this algorithm converges to an optimal ranking under a condition involving both μ and $\mathbf{P}$. For simplicity, we count here the *fraction* of times an item beats other items, which allows us to restrict our attention to conditions on $\mathbf{P}$.

Algorithm 4 BTL-ML Estimator

Input: Empirical comparison matrix $\widehat{\mathbf{P}} \in [0,1]^{n \times n}$
Find maximum likelihood estimate of BTL score vector:

$$\widehat{\boldsymbol{\theta}} \in \arg\min_{\boldsymbol{\theta} \in \mathbb{R}^n} \sum_{i<j} \Big(\ln(1+\exp(\theta_j - \theta_i)) - \widehat{P}_{ij}(\theta_j - \theta_i) \Big)$$

For $i = 1$ to n: $\widehat{w}_i = \exp(\widehat{\theta}_i)$
Output: Permutation $\widehat{\sigma} \in \text{argsort}(\widehat{\mathbf{w}})$

Theorem 16. *Let $(\mu, \mathbf{P})$ be such that $\mu_{\min} > 0$ and $\mathbf{P}$ satisfies the LN condition, and $\exists (i \neq j) : P_{ij} \neq \frac{1}{2}$. Let $\mathbf{f}^*$ be as in Eq. (10), and let $r_{\min} = \min_{i,j: f_i^* \neq f_j^*} |f_i^* - f_j^*|$. Let $\delta \in (0,1]$. If*

$$m \geq \max\left(\frac{1152}{r_{\min}^2 \mu_{\min}^2} \ln\left(\frac{4n^2}{\delta}\right),\ B(\mu_{\min})\right),$$

then with probability at least $1-\delta$ (over the random draw of $S \sim (\mu, \mathbf{P})^m$ from which $\widehat{\mathbf{P}}$ is constructed), the permutation $\widehat{\sigma}$ output by the BTL-ML algorithm satisfies

$$\widehat{\sigma} \in \text{argmin}_{\sigma \in \mathcal{S}_n} \text{er}^{\text{PD}}_{\mu,\mathbf{P}}[\sigma]\,.$$

6. Generalized Low-Noise (GLN) Condition

In this section we consider a more general condition on the preference matrix $\mathbf{P}$ that we term 'generalized low-noise':

Definition 17 (Generalized low-noise (GLN) condition). *We say the pairwise preference matrix $\mathbf{P} \in [0,1]^{n \times n}$ satisfies the* generalized low-noise (GLN) condition *if $\exists \boldsymbol{\alpha} \in \mathbb{R}^n$ such that*

$$\forall i \neq j : \quad P_{ij} > P_{ji} \implies \sum_{k=1}^n \alpha_k P_{kj} > \sum_{k=1}^n \alpha_k P_{ki}\,.$$

Clearly, the LN condition of Section 5 is a special case with $\alpha_k = 1\ \forall k \in [n]$. Moreover, if $\mathbf{P}$ satisfies the GLN condition for some vector $\boldsymbol{\alpha}$, then any permutation that ranks items in decreasing order of scores $f_i = \sum_{k=1}^n \alpha_k P_{ki}$ is an optimal permutation w.r.t. the pairwise disagreement error.

As our experiments will show, none of the four common rank aggregation algorithms considered in Sections 4–5 above are guaranteed to converge to an optimal ranking under a general probability distribution satisfying the GLN condition. Below we propose a new SVM-based rank aggregation algorithm which satisfies this property.

6.1. New Algorithm: SVM-RankAggregation

We will need the following definition:

Definition 18 ($\mathbf{P}$-Induced Dataset). *For any matrix $\mathbf{P} \in [0,1]^{n \times n}$, define the* $\mathbf{P}$-induced dataset $S_{\mathbf{P}} = \{\mathbf{v}_{ij}, z_{ij}\}_{i<j}$ *as consisting of the $\binom{n}{2}$ vectors $\mathbf{v}_{ij} = (\mathbf{P}_i - \mathbf{P}_j) \in \mathbb{R}^n$ $(i < j)$, where $\mathbf{P}_i$ denotes the i-th column of $\mathbf{P}$, together with binary labels $z_{ij} = \text{sign}(P_{ji} - P_{ij}) \in \{\pm 1\}$.*

Algorithm 5 SVM-RankAggregation

Input: Empirical comparison matrix $\widehat{\mathbf{P}} \in [0,1]^{n\times n}$.
Construct $\widehat{\mathbf{P}}$-induced dataset $S_{\widehat{\mathbf{P}}}$ (see Section 6.1)

If $S_{\widehat{\mathbf{P}}}$ is linearly separable by hyperplane through origin, then
  train hard-margin linear SVM on $S_{\widehat{\mathbf{P}}}$; obtain $\widehat{\boldsymbol{\alpha}} \in \mathbb{R}^n$
else
  train soft-margin linear SVM (with any suitable value for regularization parameter) on $S_{\widehat{\mathbf{P}}}$; obtain $\widehat{\boldsymbol{\alpha}} \in \mathbb{R}^n$

For $i = 1$ to n: $\widehat{f}_i = \sum_{k=1}^n \widehat{\alpha}_k \widehat{P}_{ki}$
Output: Permutation $\widehat{\sigma} \in \text{argsort}(\widehat{\mathbf{f}})$

Proposition 19. *Let $\mathbf{P} \in [0,1]^{n\times n}$ be a preference matrix with $P_{ij} \neq \frac{1}{2}\ \forall i,j$. Then $\mathbf{P}$ satisfies the GLN condition if and only if the $\mathbf{P}$-induced dataset $S_{\mathbf{P}} = \{\mathbf{v}_{ij}, z_{ij}\}_{i<j}$ is linearly separable by a hyperplane passing through the origin, i.e. $\exists \boldsymbol{\alpha} \in \mathbb{R}^n$ s.t. $z_{ij}\boldsymbol{\alpha}^\top \mathbf{v}_{ij} > 0\ \forall i<j$.*

While the above proposition guarantees that a preference matrix $\mathbf{P}$ satisfying the GLN condition induces a linearly separable dataset $S_{\mathbf{P}}$, in general, the dataset $S_{\widehat{\mathbf{P}}}$ induced by an empirical comparison matrix $\widehat{\mathbf{P}}$ constructed from a random sample $S \sim (\mu, \mathbf{P})^m$ may not always be linearly separable. However, as we show in the proof of Theorem 20 below, for large enough m, with high probability, $S_{\widehat{\mathbf{P}}}$ is also linearly separable by a hyperplane passing through the origin. Our SVM-RankAggregation algorithm (Algorithm 5) tests whether $S_{\widehat{\mathbf{P}}}$ is linearly separable by such a classifier; if so, it trains a hard-margin linear SVM classifier that yields such a separating hyperplane through the origin.

6.2. Convergence of SVM-RankAggregation Algorithm

We now show that the SVM-RankAggregation algorithm converges to an optimal ranking under the GLN condition, and give a sample complexity bound for SVM-RankAggregation to exactly recover an optimal ranking:

Theorem 20. *Let $(\mu, \mathbf{P})$ be such that $\mu_{\min} > 0$ and $\mathbf{P}$ satisfies the GLN condition for some vector $\boldsymbol{\alpha} \in \mathbb{R}^n$, and $P_{ij} \neq \frac{1}{2}\ \forall i,j$. Let $\gamma = \min_{i,j} |P_{ij} - \frac{1}{2}|$, and let $r^{\boldsymbol{\alpha}}_{\min} = \min_{i,j} \frac{|\boldsymbol{\alpha}^T(\mathbf{P}_i - \mathbf{P}_j)|}{\|\boldsymbol{\alpha}\|_2}$. Let $\delta \in (0,1)$. If*

$$m \geq \max\left(\frac{2048\, n}{(r^{\boldsymbol{\alpha}}_{\min}\mu_{\min})^2}\log\left(\frac{16n^3}{\delta}\right), \frac{128}{\gamma^2\mu^2_{\min}}\log\left(\frac{8n^2}{\delta}\right), B(\mu_{\min})\right),$$

then with probability at least $1-\delta$ (over the random draw of $S \sim (\mu, \mathbf{P})^m$ from which $\widehat{\mathbf{P}}$ is constructed), the permutation $\widehat{\sigma}$ output by SVM-RankAggregation satisfies

$$\widehat{\sigma} \in \text{argmin}_{\sigma \in \mathcal{S}_n} \text{er}^{\text{PD}}_{\mu,\mathbf{P}}[\sigma]\,.$$

7. Experiments

In this section we report results of experiments designed to verify our convergence results and investigate the tightness of the corresponding sample complexity bounds.

7.1. Convergence under BTL

Our first experiment was with BTL distributions for $n = 5, 10, 20$. For each n, we constructed $\mathbf{P}$ using a random BTL vector $\mathbf{w} \in \mathbb{R}^n_+$ (each component w_i chosen uniformly at random from $[0,1]$), taking μ to be the uniform distribution over the $\binom{n}{2}$ item pairs, and generated 100 random samples from $(\mu, \mathbf{P})$ for each of several sample sizes m. We then ran the 5 algorithms analyzed in Sections 4-6 on the generated samples, and for each n and m, computed the fraction of times an optimal permutation was recovered by each algorithm. The results are shown in Figure 2; as can be seen, for sufficiently large sample size, all 5 algorithms recover an optimal permutation with high probability. Similar results were obtained with non-uniform μ.

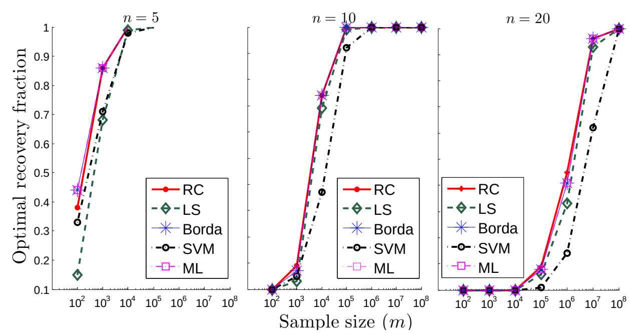

Figure 2. Fraction of times an optimal ranking was recovered by various algorithms under a BTL distribution (out of 100 random runs), for increasing sample sizes m, and for different numbers of items n (left to right: $n = 5, 10, 20$).

7.2. Convergence under LN

For our next experiment, we constructed a distribution that satisfies the LN condition but not the BTL condition; the preference matrix $\mathbf{P}$ we used (with $n = 4$) is shown below:

$$\mathbf{P} = \begin{bmatrix} 0 & 0.8 & 0.51 & 0.51 \\ 0.2 & 0 & 0.9 & 0.7 \\ 0.49 & 0.1 & 0 & 0.65 \\ 0.49 & 0.3 & 0.35 & 0 \end{bmatrix}.$$

In this case we used a random distribution μ over the item pairs (specifically, $\binom{n}{2}$ numbers u_{ij} were each chosen uniformly at random from $[0,1]$, and then normalized to yield $\mu_{ij} = u_{ij}/\sum_{kl} u_{kl}$). Again, we generated 100 random samples from $(\mu, \mathbf{P})$ for each of several sample sizes m, ran the 5 algorithms on these samples, and in each case computed the fraction of times an optimal permutation was recovered. The results are shown in Figure 3 (left); as can be seen, the rank centrality and least squares algorithms fail to recover an optimal permutation under this distribution.

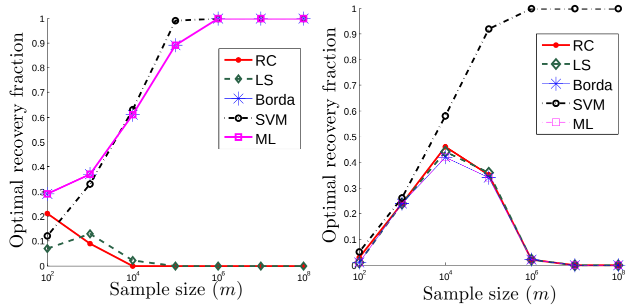

Figure 3. **Left:** Fraction of times an optimal ranking was recovered by various algorithms under a distribution satisfying the LN but not the BTL condition (out of 100 random runs), for increasing sample sizes m (here $n = 4$). **Right:** Fraction of times an optimal ranking was recovered by various algorithms under a distribution satisfying the GLN but not the LN condition (out of 100 random runs), for increasing sample sizes m (here $n = 5$).

7.3. Convergence under GLN

For our third experiment, we constructed a distribution that satisfies the GLN condition but not the LN condition; the preference matrix $\mathbf{P}$ we used (with $n = 5$) is shown below:

$$\mathbf{P} = \begin{bmatrix} 0 & 0.51 & 0.46 & 0.4 & 0.4 \\ 0.49 & 0 & 0.49 & 0.4 & 0.4 \\ 0.54 & 0.51 & 0 & 0.4 & 0.4 \\ 0.6 & 0.6 & 0.6 & 0 & 0.4 \\ 0.6 & 0.6 & 0.6 & 0.6 & 0 \end{bmatrix}.$$

It can be verified that $\mathbf{P}$ satisfies the GLN condition with $\boldsymbol{\alpha} = (-0.4530, -2.6021, 1.4660, 3.0796, 3.7197)^\top$; however $\mathbf{P}$ does not satisfy the LN condition since $P_{12} > P_{21}$ but $\sum_k P_{k1} > \sum_k P_{k2}$. Again we used a random distribution μ over the item pairs as above. The results are shown in Figure 3 (right); here only the SVM-RankAggregation algorithm successfully recovers an optimal ranking.

7.4. Tightness of Sample Complexity Bounds

Our final set of experiments was designed to evaluate the tightness of our sample complexity bounds. We first used BTL distributions generated similarly as described in Section 7.1 for various n between 5 and 20, and evaluated both the actual number of samples required by each algorithm to recover an optimal ranking at least 95% of the time, and the corresponding upper bounds on sample complexity, as a function of n. The results are shown in Figure 4 (left); in most cases, the shapes of the upper bounds are largely similar to those of the actual sample complexity curves.

Figure 4 (left) also suggests the upper bound for the rank centrality algorithm is significantly looser than those for other algorithms. This bound, which builds on techniques of (Negahban et al., 2012), involves an additional $\left(\frac{\pi_{\max}}{\pi_{\min}}\right)^3$ term not present in the other bounds. To investigate this, we designed BTL distributions for $n = 5$ with increasing values of $\left(\frac{\pi_{\max}}{\pi_{\min}}\right)$ (keeping the $r_{\min}$ term corresponding to the LN bounds in Section 5 fixed), and evaluated the sample complexity and corresponding upper bounds as a function of this ratio. The results are shown in Figure 4 (right). As can be seen, the dependence of the rank centrality upper bound on this term appears to be superfluous, and likely an artefact of the current analysis technique, which is based on that of (Negahban et al., 2012). In future work, we intend to explore alternative techniques for obtaining a tighter bound for the rank centrality algorithm.[3] We also plan to explore whether the additional factor of n in the rank centrality and SVM-RankAggregation bounds can be removed.

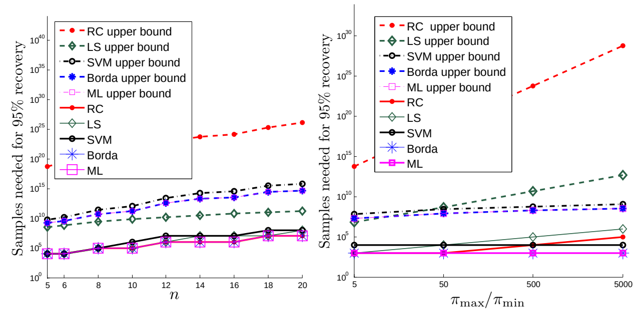

Figure 4. **Left:** Sample size required for 95% probability of recovery of an optimal ranking by various algorithms under a BTL distribution (out of 100 random runs), together with the corresponding sample complexity upper bounds, as a function of n. **Right:** Sample size required for 95% probability of recovery of an optimal ranking by various algorithms under a BTL distribution (out of 100 random runs), together with corresponding sample complexity upper bounds, as a function of $\left(\frac{\pi_{\max}}{\pi_{\min}}\right)$ (keeping $r_{\min}$ term corresponding to LN bounds constant) (here $n = 5$).

8. Conclusion

The problem of rank aggregation from pairwise comparison data has received much interest recently. We have analyzed various algorithms for this problem, and have shown that under a natural statistical model, where pairwise comparisons are drawn randomly and independently from some underlying probability distribution, the rank centrality and least squares algorithms converge to an optimal ranking under a BTL condition, while the Borda count and BTL-ML algorithms converge to an optimal ranking under a more general LN condition. However, none of these existing algorithms converges under the more general GLN condition; we have proposed a new SVM-based rank aggregation algorithm for which such convergence is guaranteed. Future work includes improving the analysis to obtain tighter bounds, and extending the analysis to other algorithms.

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[3] We note that the least squares sample complexity also shows a slight dependence on the $\left(\frac{\pi_{\max}}{\pi_{\min}}\right)$ term; this is due to the fact that this term is connected to $P_{\min}$, the dependence on which appears to be captured correctly in our bound for least squares.

References

Ailon, Nir. Aggregation of partial rankings, *p*-ratings and top-*m* lists. *Algorithmica*, 57(2):284–300, 2010.

Ailon, Nir. Active learning ranking from pairwise preferences with almost optimal query complexity. In *Advances in Neural Information Processing Systems*. 2011.

Ailon, Nir, Charikar, Moses, and Newman, Alantha. Aggregating inconsistent information: Ranking and clustering. *Journal of the ACM*, 55(5):23, 2008.

Borda, Jean-Charles de. Mémoire sur les élections au scrutin. *Histoire de l'Académie Royale des Sciences*, 1781.

Condorcet, Nicolas de. Essai sur l'application de l'analyse á la probabilité des décisions rendues á la pluralité des voix. *Imprimerie Royale, Paris*, 1785.

Cossock, David and Zhang, Tong. Statistical analysis of Bayes optimal subset ranking. *IEEE Transactions on Information Theory*, 54(11):5140–5154, 2008.

Duchi, John, Mackey, Lester, and Jordan, Michael I. On the consistency of ranking algorithms. In *Proceedings of the 27th International Conference on Machine Learning*, 2010.

Dwork, Cynthia, Kumar, Ravi, Naor, Moni, and Sivakumar, D. Rank aggregation methods for the web. In *Proceedings of the 10th International World Wide Web Conference*, 2001.

Freund, Yoav, Iyer, Raj, Schapire, Robert E., and Singer, Yoram. An efficient boosting algorithm for combining preferences. *Journal of Machine Learning Research*, 4: 933–969, 2003.

Gleich, David F. and Lim, Lek-Heng. Rank aggregation via nuclear norm minimization. In *Proceedings of the 17th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, 2011.

Guiver, John and Snelson, Edward. Bayesian inference for Plackett-Luce ranking models. In *Proceedings of the 26th International Conference on Machine Learning*, 2009.

Herbrich, R., Graepel, T., and Obermayer, K. Large margin rank boundaries for ordinal regression. *Advances in Large Margin Classifiers*, pp. 115–132, 2000.

Hochbaum, Dorit S. Ranking sports teams and the inverse equal paths problem. In *Proceedings of the 2nd International Workshop on Internet and Network Economics*, 2006.

Jagabathula, Srikanth and Shah, Devavrat. Inferring rankings under constrained sensing. In *Advances in Neural Information Processing Systems*, pp. 753–760, 2008.

Jamieson, Kevin G. and Nowak, Rob. Active ranking using pairwise comparisons. In *Advances in Neural Information Processing Systems 24*, pp. 2240–2248. 2011.

Jiang, Xiaoye, Lim, Lek-Heng, Yao, Yuan, and Ye, Yinyu. Statistical ranking and combinatorial Hodge theory. *Mathematical Programming*, 127(1):203–244, 2011.

Joachims, T. Optimizing search engines using clickthrough data. In *Proceedings of the 8th ACM Conference on Knowledge Discovery and Data Mining*, 2002.

Klementiev, Alexandre, Roth, Dan, and Small, Kevin. Unsupervised rank aggregation with distance-based models. In *Proceedings of the 25th International Conference on Machine Learning*, 2008.

Kutin, Samuel. Extensions to McDiarmid's inequality when differences are bounded with high probability. Technical report, University of Chicago, 2002.

Liu, Tie-Yan. *Learning to Rank for Information Retrieval*. Springer, 2011.

Meila, Marina, Phadnis, Kapil, Patterson, Arthur, and Bilmes, Jeff A. Consensus ranking under the exponential model. In *Proceedings of the 23rd Conference on Uncertainty in Artificial Intelligence*, 2007.

Negahban, Sahand, Oh, Sewoong, and Shah, Devavrat. Iterative ranking from pair-wise comparisons. In *Advances in Neural Information Processing Systems*, 2012.

Osting, Braxton, Brune, Christoph, and Osher, Stanley. Enhanced statistical rankings via targeted data collection. In *Proceedings of the 30th International Conference on Machine Learning*, 2013.

Qin, Tao, Geng, Xiubo, and Liu, Tie-Yan. A new probabilistic model for rank aggregation. In *Advances in Neural Information Processing Systems*, 2010.

Soufiani, Hossein Azari, Parkes, David C., and Xia, Lirong. Random utility theory for social choice. In *Advances in Neural Information Processing Systems*, 2012.

Volkovs, Maksims N. and Zemel, Richard S. A flexible generative model for preference aggregation. In *Proceedings of the 21st International World Wide Web Conference*, 2012.

Wauthier, Fabian, Jordan, Michael, and Jojic, Nebojsa. Efficient ranking from pairwise comparisons. In *Proceedings of the 30th International Conference on Machine Learning*, 2013.

Supplementary Material

Proof of Lemma 3

Proof.

Part 1. Let $m \geq \frac{4}{\mu_{\min}}$. Recall the definitions of $\widehat{P}_{ij}$, N_{ij}, $N_{ij}^{(1)}$ from Eq. (3). In the following, we will make the dependence of these quantities on the training sample explicit; specifically, for any $\omega \in (\mathcal{X} \times \{0,1\})^m$, we will write the corresponding quantities as $\widehat{P}_{ij}(\omega)$, $N_{ij}(\omega)$, and $N_{ij}^{(1)}(\omega)$, respectively.

Clearly, for any $\omega, \omega' \in (\mathcal{X} \times \{0,1\})^m$, since $\widehat{P}_{ij}(\omega), \widehat{P}_{ij}(\omega') \in [0,1]$, we have

$$|\widehat{P}_{ij}(\omega) - \widehat{P}_{ij}(\omega')| \leq 1 \,.$$

We will prove the result for the case $i < j$; the case $i > j$ can be proved similarly. Assume $i < j$, and let B_{ij} be the following 'bad' event:

$$B_{ij} = \Big\{ \omega \in (\mathcal{X} \times \{0,1\})^m : N_{ij}(\omega) \leq \tfrac{m\mu_{ij}}{2} \Big\} \,.$$

Then by a straightforward application of Hoeffding's inequality, we have

$$\mathbf{P}(S \in B_{ij}) \;\leq\; \exp(-m\mu_{ij}^2/2) \;\leq\; \exp(-m\mu_{\min}^2/2) \,.$$

Now consider $\omega, \omega' \in (\mathcal{X} \times \{0,1\})^m$ such that $\omega \notin B_{ij}$, and ω, ω' differ only in one element. We can have the following cases:

(1) $N_{ij}(\omega') = N_{ij}(\omega)$ and $N_{ij}^{(1)}(\omega') = N_{ij}^{(1)}(\omega)$

(2) $N_{ij}(\omega') = N_{ij}(\omega)$ and $N_{ij}^{(1)}(\omega') = N_{ij}^{(1)}(\omega) + 1$

(3) $N_{ij}(\omega') = N_{ij}(\omega)$ and $N_{ij}^{(1)}(\omega') = N_{ij}^{(1)}(\omega) - 1$

(4) $N_{ij}(\omega') = N_{ij}(\omega) + 1$ and $N_{ij}^{(1)}(\omega') = N_{ij}^{(1)}(\omega) + 1$

(5) $N_{ij}(\omega') = N_{ij}(\omega) + 1$ and $N_{ij}^{(1)}(\omega') = N_{ij}^{(1)}(\omega)$

(6) $N_{ij}(\omega') = N_{ij}(\omega) - 1$ and $N_{ij}^{(1)}(\omega') = N_{ij}^{(1)}(\omega) - 1$

(7) $N_{ij}(\omega') = N_{ij}(\omega) - 1$ and $N_{ij}^{(1)}(\omega') = N_{ij}^{(1)}(\omega)$

We will consider each of these cases separately, and will show that in each case, the difference $|\widehat{P}_{ij}(\omega) - \widehat{P}_{ij}(\omega')|$ is upper bounded by $\frac{2}{m\mu_{\min}}$.

- Case (1): $N_{ij}(\omega') = N_{ij}(\omega)$ and $N_{ij}^{(1)}(\omega') = N_{ij}^{(1)}(\omega)$

  In this case nothing changes with respect to the pair (i,j) and hence

$$|\widehat{P}_{ij}(\omega) - \widehat{P}_{ij}(\omega')| = 0$$

- Case (2): $N_{ij}(\omega') = N_{ij}(\omega)$ and $N_{ij}^{(1)}(\omega') = N_{ij}^{(1)}(\omega) + 1$

  In this case we have

$$\begin{aligned} |\widehat{P}_{ij}(\omega) - \widehat{P}_{ij}(\omega')| &= \left| \frac{N_{ij}^{(1)}(\omega)}{N_{ij}(\omega)} - \frac{N_{ij}^{(1)}(\omega) + 1}{N_{ij}(\omega)} \right| \\ &= \frac{1}{N_{ij}(\omega)} \\ &\leq \frac{2}{m\mu_{ij}} \leq \frac{2}{m\mu_{\min}} \end{aligned}$$

– Case (3): $N_{ij}(\omega') = N_{ij}(\omega)$ and $N_{ij}^{(1)}(\omega') = N_{ij}^{(1)}(\omega) - 1$

In this case we have

$$\begin{aligned}|\widehat{P}_{ij}(\omega) - \widehat{P}_{ij}(\omega')| &= \left|\frac{N_{ij}^{(1)}(\omega)}{N_{ij}(\omega)} - \frac{N_{ij}^{(1)}(\omega) - 1}{N_{ij}(\omega)}\right| \\ &= \frac{1}{N_{ij}(\omega)} \leq \frac{2}{m\mu_{ij}} \leq \frac{2}{m\mu_{\min}}\end{aligned}$$

– Case (4): $N_{ij}(\omega') = N_{ij}(\omega) + 1$ and $N_{ij}^{(1)}(\omega') = N_{ij}^{1}(\omega) + 1$

In this case we have

$$\begin{aligned}|\widehat{P}_{ij}(\omega) - \widehat{P}_{ij}(\omega')| &= \left|\frac{N_{ij}^{(1)}(\omega)}{N_{ij}(\omega)} - \frac{N_{ij}^{(1)}(\omega) + 1}{N_{ij}(\omega) + 1}\right| \\ &= \left|\frac{N_{ij}(\omega) - N_{ij}^{(1)}(\omega)}{N_{ij}(\omega)(N_{ij}(\omega) + 1)}\right| \leq \left|\frac{N_{ij}(\omega)}{N_{ij}(\omega)(N_{ij}(\omega) + 1)}\right| \\ &\leq \frac{1}{N_{ij}(\omega)} \leq \frac{2}{m\mu_{ij}} \leq \frac{2}{m\mu_{\min}}\end{aligned}$$

– Case (5): $N_{ij}(\omega') = N_{ij}(\omega) + 1$ and $N_{ij}^{(1)}(\omega') = N_{ij}^{(1)}(\omega)$

In this case we have

$$\begin{aligned}|\widehat{P}_{ij}(\omega) - \widehat{P}_{ij}(\omega')| &= \left|\frac{N_{ij}^{(1)}(\omega)}{N_{ij}(\omega)} - \frac{N_{ij}^{(1)}(\omega)}{N_{ij}(\omega) + 1}\right| \\ &= \left|\frac{N_{ij}^{(1)}(\omega)}{N_{ij}(\omega)(N_{ij}(\omega) + 1)}\right| \leq \left|\frac{N_{ij}(\omega)}{N_{ij}(\omega)(N_{ij}(\omega) + 1)}\right| \\ &\leq \frac{1}{N_{ij}(\omega)} \leq \frac{2}{m\mu_{ij}} \leq \frac{2}{m\mu_{\min}}\end{aligned}$$

– Case (6): $N_{ij}(\omega') = N_{ij}(\omega) - 1$ and $N_{ij}^{(1)}(\omega') = N_{ij}^{(1)}(\omega) - 1$

In this case we have

$$\begin{aligned}|\widehat{P}_{ij}(\omega) - \widehat{P}_{ij}(\omega')| &= \left|\frac{N_{ij}^{(1)}(\omega)}{N_{ij}(\omega)} - \frac{N_{ij}^{(1)}(\omega) - 1}{N_{ij}(\omega) - 1}\right| \\ &= \left|\frac{N_{ij}(\omega) - N_{ij}^{(1)}(\omega)}{N_{ij}(\omega)(N_{ij}(\omega) - 1)}\right| \\ &\leq \frac{1}{N_{ij}(\omega)} \leq \frac{2}{m\mu_{ij}} \leq \frac{2}{m\mu_{\min}}\end{aligned}$$

Note that in this case $N_{ij}(\omega) - 1$ cannot equal 0 because $m \geq \frac{4}{\mu_{\min}}$ which guarantees that for $\omega \notin B_{ij}$, $N_{ij}(\omega) \geq 2$. Also the final step follows because this case can happen only when $N_{ij}^{(1)}(\omega) \geq 1$ and so we can upper bound $\frac{(N_{ij}(\omega) - N_{ij}^{(1)}(\omega))}{(N_{ij}(\omega) - 1)}$ by 1

– Case (7): $N_{ij}(\omega') = N_{ij}(\omega) - 1$ and $N_{ij}^{(1)}(\omega') = N_{ij}^{(1)}(\omega)$

In this case we have

$$\begin{aligned}|\widehat{P}_{ij}(\omega) - \widehat{P}_{ij}(\omega')| &= \left|\frac{N^{(1)}_{ij}(\omega)}{N_{ij}(\omega)} - \frac{N^{(1)}_{ij}(\omega)}{N_{ij}(\omega)-1}\right| \\ &= \left|\frac{N^{(1)}_{ij}(\omega)}{N_{ij}(\omega)(N_{ij}(\omega)-1)}\right| \\ &\le \frac{1}{N_{ij}(\omega)} \le \frac{2}{m\mu_{ij}} \le \frac{2}{m\mu_{\min}}\end{aligned}$$

Again, $N_{ij}(\omega) - 1$ cannot equal 0 because $m \ge \frac{4}{\mu_{\min}}$ which guarantees that for $\omega \notin B_{ij}$, $N_{ij}(\omega) \ge 2$. Also note that this case can occur only when $N^{(1)}_{ij}(\omega) \le N_{ij}(\omega) - 1$ which is used to upper bound $\frac{N^{(1)}_{ij}}{N_{ij}(\omega)-1}$ by 1.

Thus we have the required bound in all possible cases.

Part 2. This follows directly from Part 1 and Theorem 2.

Part 3. Let $m \ge \frac{1}{\mu_{\min}} \ln\left(\frac{1}{\epsilon}\right)$. We have,

$$\mathbf{E}[\widehat{P}_{ij}] \;=\; P_{ij}(1-(1-\mu_{ij})^m)\,.$$

This gives

$$\left|\mathbf{E}[\widehat{P}_{ij}] - P_{ij}\right| \;=\; P_{ij}(1-\mu_{ij})^m \;\le\; (1-\mu_{\min})^m \;\le\; e^{-m\mu_{\min}} \;\le\; \epsilon\,,$$

where the last inequality follows from the given condition on m.

Part 4. Let m satisfy the given condition. Then

$$\begin{aligned}\mathbf{P}\Big(\big|\widehat{P}_{ij} - P_{ij}\big| \ge \epsilon\Big) \;&\le\; \mathbf{P}\Big(\big|\widehat{P}_{ij} - \mathbf{E}[\widehat{P}_{ij}]\big| + \big|\mathbf{E}[\widehat{P}_{ij}] - P_{ij}\big| \ge \epsilon\Big)\,, \quad \text{by triangle inequality} \\ &\le\; \mathbf{P}\Big(\big|\widehat{P}_{ij} - \mathbf{E}[\widehat{P}_{ij}]\big| \ge \frac{\epsilon}{2}\Big)\,, \quad \text{by Part 3, since } m \ge \tfrac{1}{\mu_{\min}}\ln\left(\tfrac{2}{\epsilon}\right) \\ &\le\; 4\exp\left(\frac{-m\epsilon^2\mu_{\min}^2}{128}\right)\,, \quad \text{by Part 2.}\end{aligned}$$

Part 5. Let $m \ge \frac{1}{\mu_{\min}P_{\min}} \ln\left(\frac{n(n-1)}{\delta}\right)$. Then

$$\begin{aligned}\mathbf{P}\Big(\exists(i \ne j) : \widehat{P}_{ij} = 0\Big) \;&\le\; \sum_{i=1}^{n}\sum_{j\ne i}\mathbf{P}\big(\widehat{P}_{ij} = 0\big)\,, \quad \text{by union bound} \\ &=\; \sum_{i=1}^{n}\sum_{j\ne i}(1-\mu_{ij}P_{ij})^m \\ &\le\; n(n-1)\big(1-\mu_{\min}P_{\min}\big)^m \\ &\le\; n(n-1)\,e^{-m\mu_{\min}P_{\min}} \\ &\le\; \delta\,,\end{aligned}$$

where the last inequality follows from the given condition on m.

This completes the proof of the lemma. □

Proof of Lemma 6

Proof. We will first show the forward direction. Assume that the preference matrix $\mathbf{P}$ satisfies the time-reversibility condition. Let $\mathbf{Q}$ be the time-reversible Markov chain corresponding to $\mathbf{P}$, with stationary distribution $\boldsymbol{\pi}$; since $\mathbf{Q}$ is irreducible and aperiodic, we have $\pi_i > 0\ \forall i$. Now let $i \neq j$. By time reversibility and definition of Q_{ij},

$$\pi_i P_{ij} = \pi_j P_{ji}\,.$$

We also have

$$P_{ji} = 1 - P_{ij}\,.$$

Solving for P_{ij}, this gives

$$P_{ij} = \frac{\pi_j}{\pi_i + \pi_j}\,.$$

Thus $\mathbf{P}$ satisfies the BTL condition with vector $\mathbf{w} = \boldsymbol{\pi} \in \mathbb{R}^n_+$. This proves the forward direction.

To show the reverse direction, assume that the preference matrix $\mathbf{P}$ satisfies the BTL condition with vector $\mathbf{w} \in \mathbb{R}^n_+$, so that $w_i > 0\ \forall i$ and $P_{ij} = \frac{w_j}{w_i + w_j}\ \forall i \neq j$. Let $\mathbf{Q}$ be the Markov chain constructed from $\mathbf{P}$ as in Eq. (6). Then it is easy to see that the vector $\boldsymbol{\pi}$ given by $\pi_i = \frac{w_i}{\sum_{k=1}^n w_k}$ satisfies

$$\pi_i Q_{ij} = \pi_j Q_{ji}\ \ \forall i, j \in [n]\,,$$

from which it follows that $\boldsymbol{\pi}$ is also the stationary probability vector of $\mathbf{Q}$. Therefore $\mathbf{P}$ satisfies the time-reversibility condition, thus proving the reverse direction. □

Proof of Theorem 7

The proof of Theorem 7 builds on techniques of (Negahban et al., 2012). We first state below four lemmas that are used in the proof: two of these are due to Negahban et al. (Negahban et al., 2012); proofs for the remaining two are included below. The statements of the lemmas and corresponding proofs require some additional notation as summarized below:

Additional notation. In what follows, for a matrix $\mathbf{Q} \in \mathbb{R}^{n\times n}$, we will denote by $\|\mathbf{Q}\|_F = \big(\sum_{i=1}^n \sum_{j=1}^n Q_{ij}^2\big)^{1/2}$ the Frobenius norm of $\mathbf{Q}$, by $\|\mathbf{Q}\|_2 = \max_{\mathbf{x}\in\mathbb{R}^n, \mathbf{x}\neq \mathbf{0}} \frac{\|\mathbf{Qx}\|_2}{\|\mathbf{x}\|_2}$ the spectral norm of $\mathbf{Q}$, and by $\lambda_{(2)}(\mathbf{Q})$ the second-largest eigenvalue of $\mathbf{Q}$ in absolute value.

Lemma 21. *Let $(\mu, \mathbf{P})$ be such that $\mu_{\min} > 0$. Let $\mathbf{Q}$ be defined as in Eq. (6). Let $0 < \epsilon \leq 8$ and $\delta \in (0, 1]$. If*

$$m \ \geq\ \max\left(\frac{256\,n}{\epsilon^2 \mu_{\min}^2} \ln\Big(\frac{8n^2}{\delta}\Big)\,,\ B(\mu_{\min})\right)\,,$$

then with probability at least $1 - \delta$ (over the random draw of $S \sim (\mu, \mathbf{P})^m$ from which $\widehat{\mathbf{P}}$ is constructed), the empirical Markov chain $\widehat{\mathbf{Q}}$ constructed by the rank centrality algorithm satisfies

$$\|\widehat{\mathbf{Q}} - \mathbf{Q}\|_2 \leq \epsilon\,.$$

Proof of Lemma 21. Let m satisfy the given condition. We have,

$$\begin{aligned}
\big\|\mathbf{E}[\widehat{\mathbf{Q}}] - \mathbf{Q}\big\|_F^2 \ &=\ \sum_{i=1}^n \sum_{j\neq i} \big(\mathbf{E}[\widehat{Q}_{ij}] - Q_{ij}\big)^2 + \sum_{i=1}^n \big(\mathbf{E}[\widehat{Q}_{ii}] - Q_{ii}\big)^2 \\
&=\ \sum_{i=1}^n \sum_{j\neq i} \Big(\frac{1}{n}\big(\mathbf{E}[\widehat{P}_{ij}] - P_{ij}\big)\Big)^2 + \sum_{i=1}^n \Big(\frac{1}{n}\sum_{k\neq i}\big(\mathbf{E}[\widehat{P}_{ik}] - P_{ik}\big)\Big)^2 \\
&\leq\ \frac{(n-1)}{n}\Big(\frac{\epsilon}{2\sqrt{n-1}}\Big)^2 + \frac{(n-1)^2}{n}\Big(\frac{\epsilon}{2\sqrt{n-1}}\Big)^2, \\
&\qquad \text{by Lemma 3 (part 3), since } m \geq \tfrac{256n}{\epsilon^2\mu_{\min}^2}\ln\big(\tfrac{8n^2}{\delta}\big) \geq \tfrac{1}{\mu_{\min}}\ln\big(\tfrac{2\sqrt{n-1}}{\epsilon}\big) \\
&=\ (n-1)\Big(\frac{\epsilon}{2\sqrt{n-1}}\Big)^2 \\
&=\ \frac{\epsilon^2}{4}\,.
\end{aligned} \tag{10}$$

Now,

$$\begin{aligned}
\mathbf{P}\big(\|\widehat{\mathbf{Q}}-\mathbf{Q}\|_2 \geq \epsilon\big) &\leq \mathbf{P}\big(\|\widehat{\mathbf{Q}}-\mathbf{Q}\|_F \geq \epsilon\big)\,, \quad \text{since Frobenius norm upper bounds spectral norm} \\
&\leq \mathbf{P}\big(\big\|\widehat{\mathbf{Q}}-\mathbf{E}[\widehat{\mathbf{Q}}]\big\|_F + \big\|\mathbf{E}[\widehat{\mathbf{Q}}]-\mathbf{Q}\big\|_F \geq \epsilon\big)\,, \quad \text{by triangle inequality} \\
&\leq \mathbf{P}\Big(\big\|\widehat{\mathbf{Q}}-\mathbf{E}[\widehat{\mathbf{Q}}]\big\|_F \geq \frac{\epsilon}{2}\Big)\,, \quad \text{by Eq. (10)} \\
&= \mathbf{P}\Big(\big\|\widehat{\mathbf{Q}}-\mathbf{E}[\widehat{\mathbf{Q}}]\big\|_F^2 \geq \frac{\epsilon^2}{4}\Big) \\
&= \mathbf{P}\bigg(\sum_{i=1}^n\sum_{j\neq i}\big(\widehat{Q}_{ij}-\mathbf{E}[\widehat{Q}_{ij}]\big)^2 + \sum_{i=1}^n\big(\widehat{Q}_{ii}-\mathbf{E}[\widehat{Q}_{ii}]\big)^2 \geq \frac{\epsilon^2}{4}\bigg) \\
&\leq \mathbf{P}\bigg(\sum_{i=1}^n\sum_{j\neq i}\big(\widehat{Q}_{ij}-\mathbf{E}[\widehat{Q}_{ij}]\big)^2 \geq \frac{\epsilon^2}{8}\bigg) + \mathbf{P}\bigg(\sum_{i=1}^n\big(\widehat{Q}_{ii}-\mathbf{E}[\widehat{Q}_{ii}]\big)^2 \geq \frac{\epsilon^2}{8}\bigg) \\
&\leq \sum_{i=1}^n\sum_{j\neq i}\mathbf{P}\bigg(\big|\widehat{Q}_{ij}-\mathbf{E}[\widehat{Q}_{ij}]\big| \geq \frac{\epsilon}{(\sqrt{8})\,n}\bigg) + \sum_{i=1}^n\mathbf{P}\bigg(\big|\widehat{Q}_{ii}-\mathbf{E}[\widehat{Q}_{ii}]\big| \geq \frac{\epsilon}{\sqrt{8n}}\bigg) \\
&= \sum_{i=1}^n\sum_{j\neq i}\mathbf{P}\bigg(\frac{1}{n}\big|\widehat{P}_{ij}-\mathbf{E}[\widehat{P}_{ij}]\big| \geq \frac{\epsilon}{(\sqrt{8})\,n}\bigg) + \sum_{i=1}^n\mathbf{P}\bigg(\frac{1}{n}\bigg|\sum_{k\neq i}\big(\widehat{P}_{ik}-\mathbf{E}[\widehat{P}_{ik}]\big)\bigg| \geq \frac{\epsilon}{\sqrt{8n}}\bigg) \\
&\leq \sum_{i=1}^n\sum_{j\neq i}\mathbf{P}\bigg(\big|\widehat{P}_{ij}-\mathbf{E}[\widehat{P}_{ij}]\big| \geq \frac{\epsilon}{\sqrt{8}}\bigg) + \sum_{i=1}^n\mathbf{P}\bigg(\frac{1}{n}\sum_{k\neq i}\big|\widehat{P}_{ik}-\mathbf{E}[\widehat{P}_{ik}]\big| \geq \frac{\epsilon}{\sqrt{8n}}\bigg) \\
&\leq \sum_{i=1}^n\sum_{j\neq i}\mathbf{P}\bigg(\big|\widehat{P}_{ij}-\mathbf{E}[\widehat{P}_{ij}]\big| \geq \frac{\epsilon}{\sqrt{8}}\bigg) + \sum_{i=1}^n\sum_{k\neq i}\mathbf{P}\bigg(\big|\widehat{P}_{ik}-\mathbf{E}[\widehat{P}_{ik}]\big| \geq \frac{\epsilon}{\sqrt{8n}}\bigg) \\
&\leq 4n^2\exp\bigg(\frac{-m\epsilon^2\mu_{\min}^2}{256}\bigg) + 4n^2\exp\bigg(\frac{-m\epsilon^2\mu_{\min}^2}{256n}\bigg)\,, \quad \text{by Lemma 3 (part 2)} \\
&\leq \frac{\delta}{2}+\frac{\delta}{2} \;=\; \delta\,, \quad \text{since } m \geq \tfrac{256n}{\epsilon^2\mu_{\min}^2}\ln\big(\tfrac{8n^2}{\delta}\big).
\end{aligned}$$

This proves the result. □

Lemma 22 ((Negahban et al., 2012))**.** *Let $\mathbf{Q}$ and $\widetilde{\mathbf{Q}}$ be time-reversible Markov chains defined on the same transition probability graph $G=([n],E)$, with stationary probability vectors $\boldsymbol{\pi}$ and $\widetilde{\boldsymbol{\pi}}$, respectively. Let $\alpha=\min_{(i,j)\in E}\frac{\pi_i Q_{ij}}{\widetilde{\pi}_i\widetilde{Q}_{ij}}$ and $\beta=\max_i\frac{\pi_i}{\widetilde{\pi}_i}$. Then*

$$1-\lambda_{(2)}(\mathbf{Q}) \;\geq\; \frac{\alpha}{\beta}\big(1-\lambda_{(2)}(\widetilde{\mathbf{Q}})\big)\,.$$

Lemma 23. *Let $\mathbf{Q}\in[0,1]^{n\times n}$ be the transition probability matrix of a time-reversible Markov chain with $Q_{ij}>0\;\forall i,j$, and let $\boldsymbol{\pi}$ be the stationary probability vector of $\mathbf{Q}$. Let $Q_{\min}=\min_{i,j}Q_{ij}$, $\pi_{\max}=\max_i\pi_i$, and $\pi_{\min}=\min_i\pi_i$. Then the spectral gap of $\mathbf{Q}$ satisfies*

$$1-\lambda_{(2)}(\mathbf{Q}) \;\geq\; n\left(\frac{\pi_{\min}}{\pi_{\max}}\right)Q_{\min}\,.$$

Proof of Lemma 23. Since $Q_{ij}>0\;\forall i,j$, the chain $\mathbf{Q}$ is defined on the complete directed graph $G=([n],[n]\times[n])$. Define a time-reversible Markov chain $\widetilde{\mathbf{Q}}$ on the same graph as follows:

$$\widetilde{Q}_{ij}=\frac{1}{n}\;\;\forall i,j\in[n]\,.$$

This has stationary probability vector $\widetilde{\boldsymbol{\pi}}$ given by $\widetilde{\pi}_i=\frac{1}{n}\;\forall i$. Now, using the notation of Lemma 22, we have

$$\begin{aligned}
\alpha &= n^2\pi_{\min}Q_{\min} \\
\beta &= n\,\pi_{\max}\,.
\end{aligned}$$

Moreover, $\lambda_{(2)}(\widetilde{\mathbf{Q}}) = 0$. By Lemma 22, we therefore have

$$1 - \lambda_{(2)}(\mathbf{Q}) \;\geq\; \frac{n^2 \pi_{\min} Q_{\min}}{n \pi_{\max}} \;=\; n \left(\frac{\pi_{\min}}{\pi_{\max}}\right) Q_{\min} \,.$$

□

Lemma 24 ((Negahban et al., 2012)). *Let $\mathbf{Q}$ be a time-reversible Markov chain with stationary probability vector $\boldsymbol{\pi}$. Let $\widehat{\mathbf{Q}}$ be any other Markov chain, and let $\mathbf{q}_t$ denote the state distribution of $\widehat{\mathbf{Q}}$ at time t when started with initial distribution $\mathbf{q}_0$. Let $\pi_{\max} = \max_i \pi_i$, $\pi_{\min} = \min_i \pi_i$, and $\rho = \lambda_{(2)}(\mathbf{Q}) + \|\widehat{\mathbf{Q}} - \mathbf{Q}\|_2 \sqrt{\frac{\pi_{\max}}{\pi_{\min}}}$. Then*

$$\frac{\|\mathbf{q}_t - \boldsymbol{\pi}\|_2}{\|\boldsymbol{\pi}\|_2} \;\leq\; \rho^t \frac{\|\mathbf{q}_0 - \boldsymbol{\pi}\|_2}{\|\boldsymbol{\pi}\|_2} \sqrt{\frac{\pi_{\max}}{\pi_{\min}}} + \frac{1}{1-\rho} \|\widehat{\mathbf{Q}} - \mathbf{Q}\|_2 \sqrt{\frac{\pi_{\max}}{\pi_{\min}}} \,.$$

We are now ready to prove Theorem 7:

Proof of Theorem 7. Let m satisfy the given condition. Then by Lemma 21, we have with probability at least $1 - \frac{\delta}{2}$, the empirical Markov chain $\widehat{\mathbf{Q}}$ constructed by the rank centrality algorithm satisfies

$$\|\widehat{\mathbf{Q}} - \mathbf{Q}\|_2 \;\leq\; \frac{\epsilon}{2} \left(\frac{\pi_{\min}}{\pi_{\max}}\right)^{3/2} P_{\min} \,. \tag{11}$$

In this case, since $\mathbf{Q}$ is time-reversible with $Q_{ij} > 0 \;\forall i, j$ and $Q_{\min} = \min_{i,j} Q_{ij} = \frac{P_{\min}}{n}$, by Lemma 23 and Eq. (11), we have

$$\begin{aligned}
\rho \;=\; \lambda_{(2)}(\mathbf{Q}) + \|\widehat{\mathbf{Q}} - \mathbf{Q}\|_2 \sqrt{\frac{\pi_{\max}}{\pi_{\min}}} \quad &\leq \quad 1 - \left(\frac{\pi_{\min}}{\pi_{\max}}\right) P_{\min} + \frac{\epsilon}{2} \left(\frac{\pi_{\min}}{\pi_{\max}}\right) P_{\min} \\
&\leq \quad 1 - \frac{1}{2} \left(\frac{\pi_{\min}}{\pi_{\max}}\right) P_{\min} \,.
\end{aligned} \tag{12}$$

Next, since $m \geq \frac{1024\, n}{\epsilon^2 P_{\min}^2 \mu_{\min}^2} \left(\frac{\pi_{\max}}{\pi_{\min}}\right)^3 \ln\left(\frac{16 n^2}{\delta}\right) \geq \frac{1}{\mu_{\min} P_{\min}} \ln\left(\frac{2n(n-1)}{\delta}\right)$, by Lemma 3 (part 5), we have that with probability at least $1 - \frac{\delta}{2}$, $\widehat{P}_{ij} > 0 \;\forall i \neq j$, and therefore $\widehat{\mathbf{Q}}$ is an irreducible and aperiodic Markov chain.

Putting the above two statements together, with probability at least $1-\delta$, we have that $\widehat{\mathbf{Q}}$ is an irreducible, aperiodic Markov chain satisfying Eqs. (11-12), and the score vector $\widehat{\boldsymbol{\pi}}$ output by the rank centrality algorithm is the stationary probability vector of $\widehat{\mathbf{Q}}$. In this case, by Lemma 24, we have that for any initial distribution $\mathbf{q}_0$ of $\widehat{\mathbf{Q}}$,

$$\begin{aligned}
\frac{\|\widehat{\boldsymbol{\pi}} - \boldsymbol{\pi}\|_2}{\|\boldsymbol{\pi}\|_2} \;=\; \lim_{t \to \infty} \frac{\|\mathbf{q}_t - \boldsymbol{\pi}\|_2}{\|\boldsymbol{\pi}\|_2} \quad &\leq \quad \lim_{t \to \infty} \rho^t \frac{\|\mathbf{q}_0 - \boldsymbol{\pi}\|_2}{\|\boldsymbol{\pi}\|_2} \sqrt{\frac{\pi_{\max}}{\pi_{\min}}} + \frac{1}{1-\rho} \|\widehat{\mathbf{Q}} - \mathbf{Q}\|_2 \sqrt{\frac{\pi_{\max}}{\pi_{\min}}} \\
&= \quad \frac{1}{1-\rho} \|\widehat{\mathbf{Q}} - \mathbf{Q}\|_2 \sqrt{\frac{\pi_{\max}}{\pi_{\min}}} \,, \quad \text{since } \rho < 1 \text{, by Eq. (12)} \\
&\leq \quad \epsilon \,, \quad \text{by Eqs. (11-12).}
\end{aligned}$$

The result follows since $\|\boldsymbol{\pi}\|_2 \leq 1$, which gives $\|\widehat{\boldsymbol{\pi}} - \boldsymbol{\pi}\|_2 \leq \frac{\|\widehat{\boldsymbol{\pi}} - \boldsymbol{\pi}\|_2}{\|\boldsymbol{\pi}\|_2}$. □

Proof of Corollary 8

Proof. Let $\epsilon = \frac{r_{\min}}{3}$. By definition of $r_{\min}$, we have $r_{\min} \leq 1$, and therefore $\epsilon \leq \frac{1}{3} < 1$. Therefore if m satisfies the given condition, then by Theorem 7, we have with probability at least $1 - \delta$,

$$\|\widehat{\boldsymbol{\pi}} - \boldsymbol{\pi}\|_2 \leq \frac{r_{\min}}{3} \,.$$

But

$$\begin{aligned}
\|\widehat{\boldsymbol{\pi}} - \boldsymbol{\pi}\|_2 \leq \frac{r_{\min}}{3} \quad & \Longrightarrow \quad \|\widehat{\boldsymbol{\pi}} - \boldsymbol{\pi}\|_\infty \leq \frac{r_{\min}}{3}\,, \quad \text{since } L_2 \text{ norm upper bounds } L_\infty \text{ norm} \\
& \Longrightarrow \quad |\widehat{\pi}_i - \pi_i| \leq \frac{r_{\min}}{3} \;\; \forall i \\
& \Longrightarrow \quad \Big\{ \forall i,j : \; \pi_j > \pi_i \implies \widehat{\pi}_j > \widehat{\pi}_i \Big\}\,, \quad \text{by definition of } r_{\min} \\
& \Longrightarrow \quad \Big\{ \forall i,j : \; P_{ij} > P_{ji} \implies \widehat{\pi}_j > \widehat{\pi}_i \Big\}\,, \quad \text{by time-reversibility condition on } \mathbf{P} \\
& \Longrightarrow \quad \text{argsort}(\widehat{\boldsymbol{\pi}}) \subseteq \text{argmin}_{\sigma \in \mathcal{S}_n} \text{er}^{\text{PD}}_{\mu,\mathbf{P}}[\sigma] \\
& \Longrightarrow \quad \widehat{\sigma} \in \text{argmin}_{\sigma \in \mathcal{S}_n} \text{er}^{\text{PD}}_{\mu,\mathbf{P}}[\sigma]\,.
\end{aligned}$$

Thus we have that with probability at least $1 - \delta$,

$$\widehat{\sigma} \in \text{argmin}_{\sigma \in \mathcal{S}_n} \text{er}^{\text{PD}}_{\mu,\mathbf{P}}[\sigma]\,.$$

This proves the result. □

Proof of Lemma 9

Proof. Let $\mathbf{Q}$ be as defined in Eq. (6), and let $\boldsymbol{\pi}$ be the stationary probability vector of $\mathbf{Q}$. From Section 6, we know that $\mathbf{P}$ satisfies the time-reversibility condition, and therefore any permutation that ranks items according to decreasing order of scores π_i is an optimal permutation w.r.t. the pairwise disagreement error, i.e. we have

$$\text{argsort}(\boldsymbol{\pi}) \subseteq \text{argmin}_{\sigma \in \mathcal{S}_n} \text{er}^{\text{PD}}_{\mu,\mathbf{P}}[\sigma]\,.$$

We will show that argsort$(\mathbf{f}^*) =$ argsort$(\boldsymbol{\pi})$, which will imply the result. We have,

$$\begin{aligned}
f_i^* \;=\; -\frac{1}{n}\sum_{k=1}^n Y_{ik} \;=\; \frac{1}{n}\sum_{k=1}^n \ln\left(\frac{P_{ki}}{P_{ik}}\right) \quad & = \quad \frac{1}{n}\ln\left(\prod_{k=1}^n \frac{P_{ki}}{P_{ik}}\right) \\
& = \quad \frac{1}{n}\ln\left(\prod_{k=1}^n \frac{\pi_i}{\pi_k}\right)\,, \quad \text{by time-reversibility} \\
& = \quad \ln \pi_i - \frac{1}{n}\ln(\pi_1 \cdot \ldots \cdot \pi_n)\,.
\end{aligned}$$

The second term on the right-hand side is a constant, and $\ln(\cdot)$ is a strictly monotonically increasing function; therefore $\mathbf{f}^*$ induces the same orderings as $\boldsymbol{\pi}$, i.e. argsort$(\mathbf{f}^*) =$ argsort$(\boldsymbol{\pi})$. □

Proof of Theorem 10

The proof makes use of the following technical lemma:

Lemma 25. *Let* $0 < u, u' < 1$. *Let* $0 < \epsilon < u$. *Then*

$$|u - u'| \leq \epsilon \implies \big|\ln(u) - \ln(u')\big| \leq \frac{\epsilon}{u - \epsilon}\,.$$

Proof. Let $|u - u'| \leq \epsilon$. Thus $u' \in (u - \epsilon, u + \epsilon)$. Now, since $\ln(\cdot)$ is a concave function, we have

$$\ln(y) \leq \ln(x) + \frac{1}{x}(y - x) \;\; \forall x, y > 0\,.$$

Taking $x = u$ and $y = u + \epsilon$ gives

$$\ln(u + \epsilon) \;\leq\; \ln(u) + \frac{\epsilon}{u}\,;$$

taking $x = u - \epsilon$ and $y = u$ gives

$$\ln(u) \;\leq\; \ln(u - \epsilon) + \frac{\epsilon}{u - \epsilon}\,.$$

Combining both, and using the fact that $\ln(\cdot)$ is a monotonically increasing function, we get

$$\ln(u) - \frac{\epsilon}{u-\epsilon} \;\le\; \ln(u-\epsilon) \;\le\; \ln(u) \;\le\; \ln(u+\epsilon) \;\le\; \ln(u) + \frac{\epsilon}{u}\,.$$

For $\epsilon < u$, we have $\frac{\epsilon}{u-\epsilon} > \frac{\epsilon}{u}$. Thus, since $u' \in (u-\epsilon, u+\epsilon)$, we have either

$$\ln(u) - \frac{\epsilon}{u-\epsilon} \;\le\; \ln(u-\epsilon) \;\le\; \ln(u') \;\le\; \ln(u)$$

or

$$\ln(u) \;\le\; \ln(u') \;\le\; \ln(u+\epsilon) \;\le\; \ln(u) + \frac{\epsilon}{u} \;<\; \ln(u) + \frac{\epsilon}{u-\epsilon}\,;$$

in both cases, we get $\big|\ln(u) - \ln(u')\big| \le \frac{\epsilon}{u-\epsilon}$, thus proving the result. □

Proof of Theorem 10. Let m satisfy the given condition. Since $m \ge \frac{128}{P_{\min}^2 \mu_{\min}^2}\big(1+\frac{2}{\epsilon}\big)^2 \ln\big(\frac{16n^2}{\delta}\big) \ge \frac{1}{\mu_{\min} P_{\min}} \ln\big(\frac{2n(n-1)}{\delta}\big)$, by Lemma 3 (part 5), we have with probability at least $1-\frac{\delta}{2}$, $\widehat{P}_{ij} > 0 \;\forall i \ne j$. In this case, we have

$$\widehat{Y}_{ij} = \begin{cases} \ln\Big(\frac{\widehat{P}_{ij}}{\widehat{P}_{ji}}\Big) & \text{if } i \ne j \\ 0 & \text{otherwise,} \end{cases}$$

and $\widehat{E} = \mathcal{X}$. As discussed in (Jiang et al., 2011), the score vector $\widehat{\mathbf{f}}$ output by the least squares algorithm in this case is given by

$$\widehat{f}_i \;=\; -\frac{1}{n}\sum_{k=1}^{n} \widehat{Y}_{ik} \;=\; \frac{1}{n}\sum_{k \ne i} \ln\bigg(\frac{\widehat{P}_{ki}}{\widehat{P}_{ik}}\bigg)\,.$$

Moreover, since $P_{ij} \in (0,1) \;\forall i \ne j$, we also have

$$f_i^* \;=\; =\; -\frac{1}{n}\sum_{k=1}^{n} Y_{ik} \;=\; \frac{1}{n}\sum_{k \ne i} \ln\bigg(\frac{P_{ki}}{P_{ik}}\bigg)\,.$$

Next, we have

$$\begin{aligned}
&\mathbf{P}\bigg(\exists i : \bigg|\frac{1}{n}\sum_{k\neq i}\ln\Big(\frac{\widehat{P}_{ki}}{\widehat{P}_{ik}}\Big) - \frac{1}{n}\sum_{k\neq i}\ln\Big(\frac{P_{ki}}{P_{ik}}\Big)\bigg| \geq \epsilon\bigg) \\
&\leq \sum_{i=1}^{n}\mathbf{P}\bigg(\frac{1}{n}\sum_{k\neq i}\Big|\ln\Big(\frac{\widehat{P}_{ki}}{\widehat{P}_{ik}}\Big) - \ln\Big(\frac{P_{ki}}{P_{ik}}\Big)\Big| \geq \epsilon\bigg), \quad \text{by union bound and triangle inequality} \\
&\leq \sum_{i=1}^{n}\mathbf{P}\bigg(\exists k\neq i : \Big|\ln\Big(\frac{\widehat{P}_{ki}}{\widehat{P}_{ik}} - \ln\Big(\frac{P_{ki}}{P_{ik}}\Big)\Big)\Big| \geq \epsilon\bigg), \\
&\leq \sum_{i=1}^{n}\sum_{k\neq i}\mathbf{P}\bigg(\Big|\ln\Big(\frac{\widehat{P}_{ki}}{\widehat{P}_{ik}}\Big) - \ln\Big(\frac{P_{ki}}{P_{ik}}\Big)\Big| \geq \epsilon\bigg), \quad \text{by union bound} \\
&\leq \sum_{i=1}^{n}\sum_{k\neq i}\mathbf{P}\bigg(\big|\ln \widehat{P}_{ki} - \ln P_{ki}\big| + \big|\ln \widehat{P}_{ik} - \ln P_{ik}\big| \geq \epsilon\bigg) \\
&\leq 2\sum_{i=1}^{n}\sum_{k\neq i}\mathbf{P}\bigg(\big|\ln \widehat{P}_{ki} - \ln P_{ki}\big| \geq \frac{\epsilon}{2}\bigg) \\
&\leq 2\sum_{i=1}^{n}\sum_{k\neq i}\mathbf{P}\bigg(\big|\widehat{P}_{ki} - P_{ki}\big| \geq \frac{\epsilon P_{\min}}{2+\epsilon}\bigg), \quad \text{by Lemma 25} \\
&\leq 8n^2\exp\Big(\frac{-m\epsilon^2 P_{\min}^2\mu_{\min}^2}{128(2+\epsilon)^2}\Big), \quad \text{by Lemma 3 (part 4)} \\
&\qquad\qquad \text{(since } m \geq \tfrac{128}{P_{\min}^2\mu_{\min}^2}\big(1+\tfrac{2}{\epsilon}\big)^2\ln\big(\tfrac{16n^2}{\delta}\big) \geq \tfrac{1}{\mu_{\min}}\ln\big(\tfrac{2(2+\epsilon)}{\epsilon P_{\min}}\big)) \\
&\leq \frac{\delta}{2}, \quad \text{since } m \geq \tfrac{128}{P_{\min}^2\mu_{\min}^2}\big(1+\tfrac{2}{\epsilon}\big)^2\ln\big(\tfrac{16n^2}{\delta}\big).
\end{aligned}$$

In other words, with probability at least $1-\frac{\delta}{2}$, we have

$$\max_i\bigg|\frac{1}{n}\sum_{k\neq i}\ln\Big(\frac{\widehat{P}_{ki}}{\widehat{P}_{ik}}\Big) - \frac{1}{n}\sum_{k\neq i}\ln\Big(\frac{P_{ki}}{P_{ik}}\Big)\bigg| \;\leq\; \epsilon\,.$$

Putting the above statements together, we have that with probability at least $1-\delta$,

$$\|\widehat{\mathbf{f}}-\mathbf{f}^*\|_\infty \;=\; \max_i\big|\widehat{f}_i - f_i^*\big| \;=\; \max_i\bigg|\frac{1}{n}\sum_{k\neq i}\ln\Big(\frac{\widehat{P}_{ki}}{\widehat{P}_{ik}}\Big) - \frac{1}{n}\sum_{k\neq i}\ln\Big(\frac{P_{ki}}{P_{ik}}\Big)\bigg| \;\leq\; \epsilon\,.$$

This proves the result. □

Proof of Corollary 11

Proof. Let $\epsilon = \frac{r_{\min}}{3}$. By definition of $r_{\min}$, we have $r_{\min} \leq n$, and therefore $\epsilon \leq \frac{n}{3} \leq (4\sqrt{2})n$. Therefore if m satisfies the given condition, then by Theorem 10, we have with probability at least $1-\delta$,

$$\|\widehat{\mathbf{f}}-\mathbf{f}^*\|_\infty \leq \frac{r_{\min}}{3}\,.$$

But

$$\begin{aligned}
\|\widehat{\mathbf{f}}-\mathbf{f}^*\|_\infty \leq \frac{r_{\min}}{3} \quad &\Longrightarrow \quad |\widehat{f}_i - f_i^*| \leq \frac{r_{\min}}{3} \;\; \forall i \\
&\Longrightarrow \quad \Big\{\forall i,j:\; f_j^* > f_i^* \implies \widehat{f}_j > \widehat{f}_i\Big\}, \quad \text{by definition of } r_{\min} \\
&\Longrightarrow \quad \Big\{\forall i,j:\; P_{ij} > P_{ji} \implies \widehat{f}_j > \widehat{f}_i\Big\}, \quad \text{by Lemma 9} \\
&\Longrightarrow \quad \text{argsort}(\widehat{\mathbf{f}}) \subseteq \text{argmin}_{\sigma\in\mathcal{S}_n}\text{er}^{\text{PD}}_{\mu,\mathbf{P}}[\sigma] \\
&\Longrightarrow \quad \widehat{\sigma} \in \text{argmin}_{\sigma\in\mathcal{S}_n}\text{er}^{\text{PD}}_{\mu,\mathbf{P}}[\sigma]\,.
\end{aligned}$$

Thus we have that with probability at least $1-\delta$,

$$\widehat{\sigma} \in \text{argmin}_{\sigma \in \mathcal{S}_n} \text{er}^{\text{PD}}_{\mu,\mathbf{P}}[\sigma] \,.$$

This proves the result. □

Proof of Lemma 13

Proof. Let $\mathbf{P} \in [0,1]^{n \times n}$ satisfy the BTL condition with vector $\mathbf{w} \in \mathbb{R}^n_+$, so that $w_i > 0 \; \forall i$ and $P_{ij} = \frac{w_j}{w_i + w_j} \; \forall i \neq j$. Then we have

$$\begin{aligned}
P_{ij} > P_{ji} \quad &\implies \quad w_j > w_i \\
&\implies \quad \frac{w_j}{w_j + w_k} > \frac{w_i}{w_i + w_k} \;\; \forall k \\
&\implies \quad \sum_{k=1}^{n} \frac{w_j}{w_j + w_k} > \sum_{k=1}^{n} \frac{w_i}{w_i + w_k} \\
&\implies \quad \sum_{k=1}^{n} P_{kj} > \sum_{k=1}^{n} P_{ki} \,.
\end{aligned}$$

Thus $\mathbf{P}$ satisfies the LN condition. □

Proof of Theorem 14

Proof. Let m satisfy the given condition. We have,

$$\begin{aligned}
\mathbf{P}\big(\|\widehat{\mathbf{f}} - \mathbf{f}^*\|_\infty \geq \epsilon\big) \;&=\; \mathbf{P}\big(\exists i : |\widehat{f}_i - f^*_i| \geq \epsilon\big) \\
&\leq\; \sum_{i=1}^{n} \mathbf{P}\big(|\widehat{f}_i - f^*_i| \geq \epsilon\big)\,, \quad \text{by union bound} \\
&=\; \sum_{i=1}^{n} \mathbf{P}\left(\left|\frac{1}{n}\sum_{k=1}^{n}\big(\widehat{P}_{ki} - P_{ki}\big)\right| \geq \epsilon\right), \quad \text{by definition of } \widehat{f}_i \text{ and } f^*_i \\
&\leq\; \sum_{i=1}^{n} \mathbf{P}\left(\frac{1}{n}\sum_{k=1}^{n}\big|\widehat{P}_{ki} - P_{ki}\big| \geq \epsilon\right) \\
&\leq\; \sum_{i=1}^{n} \mathbf{P}\Big(\exists k : \big|\widehat{P}_{ki} - P_{ki}\big| \geq \epsilon\Big) \\
&\leq\; \sum_{i=1}^{n}\sum_{k=1}^{n} \mathbf{P}\Big(\big|\widehat{P}_{ki} - P_{ki}\big| \geq \epsilon\Big)\,, \quad \text{by union bound} \\
&\leq\; 4n^2 \exp\left(\frac{-m\epsilon^2\mu^2_{\min}}{128}\right), \quad \text{by Lemma 3 (part 4)} \\
&\qquad\qquad \left(\text{since } m \geq \tfrac{128}{\epsilon^2\mu^2_{\min}} \ln\left(\tfrac{4n^2}{\delta}\right) \geq \tfrac{1}{\mu_{\min}} \ln\left(\tfrac{2n}{\epsilon}\right)\right) \\
&\leq\; \delta\,, \quad \text{since } m \geq \tfrac{128}{\epsilon^2\mu^2_{\min}} \ln\left(\tfrac{4n^2}{\delta}\right).
\end{aligned}$$

This proves the result. □

Proof of Corollary 15

Proof. Let $\epsilon = \frac{r_{\min}}{3}$. By definition of $r_{\min}$, we have $r_{\min} \leq n$, and therefore $\epsilon \leq \frac{n}{3} \leq (4\sqrt{2})n$. Therefore if m satisfies the given condition, then by Theorem 14, we have with probability at least $1-\delta$,

$$\|\widehat{\mathbf{f}} - \mathbf{f}^*\|_\infty \leq \frac{r_{\min}}{3} \,.$$

But

$$\|\widehat{\mathbf{f}} - \mathbf{f}^*\|_\infty \leq \frac{r_{\min}}{3} \quad\implies\quad |\widehat{f}_i - f_i^*| \leq \frac{r_{\min}}{3} \;\; \forall i$$
$$\implies\quad \Big\{\forall i,j:\; f_j^* > f_i^* \implies \widehat{f}_j > \widehat{f}_i\Big\}\,, \quad \text{by definition of } r_{\min}$$
$$\implies\quad \Big\{\forall i,j:\; P_{ij} > P_{ji} \implies \widehat{f}_j > \widehat{f}_i\Big\}\,, \quad \text{by extended low-noise condition on } \mathbf{P}$$
$$\implies\quad \text{argsort}(\widehat{\mathbf{f}}) \subseteq \text{argmin}_{\sigma\in\mathcal{S}_n} \text{er}^{\text{PD}}_{\mu,\mathbf{P}}[\sigma]$$
$$\implies\quad \widehat{\sigma} \in \text{argmin}_{\sigma\in\mathcal{S}_n} \text{er}^{\text{PD}}_{\mu,\mathbf{P}}[\sigma]\,.$$

Thus we have that with probability at least $1-\delta$,

$$\widehat{\sigma} \in \text{argmin}_{\sigma\in\mathcal{S}_n} \text{er}^{\text{PD}}_{\mu,\mathbf{P}}[\sigma]\,.$$

This proves the result. □

Proof of Theorem 16

Proof. We have

$$\widehat{\boldsymbol{\theta}} \in \arg\min_{\boldsymbol{\theta}\in\mathbb{R}^n} \sum_{i<j} \Big(\ln(1+\exp(\theta_j-\theta_i)) - \widehat{P}_{ij}(\theta_j-\theta_i) \Big)\,.$$

Setting the gradient of the above objective to $\mathbf{0}$ gives:

$$\forall i: \quad \sum_{k=1}^n \widehat{P}_{ki} \;=\; \sum_{k\neq i} \frac{\exp(\widehat{\theta}_i)}{\exp(\widehat{\theta}_k)+\exp(\widehat{\theta}_i)} \;=\; \sum_{k=1}^n P^{\widehat{\boldsymbol{\theta}}}_{ki}\,, \tag{13}$$

where we denote

$$P^{\widehat{\boldsymbol{\theta}}}_{ij} \;=\; \begin{cases} \frac{\exp(\widehat{\theta}_j)}{\exp(\widehat{\theta}_i)+\exp(\widehat{\theta}_j)} & \text{if } i<j \\ 1-P^{\widehat{\boldsymbol{\theta}}}_{ji} & \text{if } i>j \\ 0 & \text{if } i=j. \end{cases}$$

Now, we have for any $0<\epsilon<4\sqrt{2}$, if $m \geq \max\big(B(\mu_{\min}),\; \frac{1}{\mu_{\min}}\ln(\frac{2}{\epsilon})\big)$, then

$$\begin{aligned}
\mathbf{P}\Big(\exists i: \Big|\frac{1}{n}\sum_{k=1}^n \big(P_{ki}-P^{\widehat{\boldsymbol{\theta}}}_{ki}\big)\Big| \geq \epsilon\Big) &= \mathbf{P}\Big(\exists i: \Big|\frac{1}{n}\sum_{k=1}^n \big(P_{ki}-\widehat{P}_{ki}\big)\Big| \geq \epsilon\Big)\,, \quad \text{by Eq. (13)} \\
&\leq \sum_{i=1}^n \mathbf{P}\Big(\Big|\frac{1}{n}\sum_{k=1}^n \big(P_{ki}-\widehat{P}_{ki}\big)\Big| \geq \epsilon\Big) \\
&\leq \sum_{i=1}^n \mathbf{P}\Big(\frac{1}{n}\sum_{k=1}^n \big|P_{ki}-\widehat{P}_{ki}\big| \geq \epsilon\Big) \\
&\leq \sum_{i=1}^n \mathbf{P}\Big(\exists k: \big|P_{ki}-\widehat{P}_{ki}\big| \geq \epsilon\Big) \\
&\leq \sum_{i=1}^n \sum_{k=1}^n \mathbf{P}\big(\big|P_{ki}-\widehat{P}_{ki}\big| \geq \epsilon\big) \\
&\leq 4n^2 \exp\Big(\frac{-m\epsilon^2\mu_{\min}^2}{128}\Big)\,, \quad \text{by Lemma 3 (part 4).}
\end{aligned}$$

Setting $\epsilon = \frac{r_{\min}}{3}$, we get that if m satisfies the given condition, then with probability at least $1-\delta$,

$$\Big|\frac{1}{n}\sum_{k=1}^n \big(P_{kj}-P^{\widehat{\boldsymbol{\theta}}}_{kj}\big)\Big| \;\leq\; \frac{r_{\min}}{3} \quad \forall i\,. \tag{14}$$

By definition of $r_{\min}$, this means that with probability at least $1-\delta$, we have for all $i \neq j$,

$$\sum_{k=1}^{n} P^{\widehat{\boldsymbol{\theta}}}_{kj} > \sum_{k=1}^{n} P^{\widehat{\boldsymbol{\theta}}}_{ki} \iff \sum_{k=1}^{n} P_{kj} > \sum_{k=1}^{n} P_{ki} \iff f^*_j > f^*_i \,. \tag{15}$$

Also, it is easy to verify that for all $i \neq j$,

$$\widehat{w}_j > \widehat{w}_i \iff \widehat{\theta}_j > \widehat{\theta}_i \iff \sum_{k=1}^{n} P^{\widehat{\boldsymbol{\theta}}}_{kj} > \sum_{k=1}^{n} P^{\widehat{\boldsymbol{\theta}}}_{ki} \tag{16}$$

Combining Eqs. (15-16), we have that with probability at least $1-\delta$,

$$\text{argsort}(\widehat{\mathbf{w}}) = \text{argsort}(\mathbf{f}^*) \,.$$

Since $\text{argsort}(\mathbf{f}^*) \subseteq \text{argmin}_{\sigma \in \mathcal{S}_n} \text{er}^{\text{PD}}_{\mu,\mathbf{P}}[\sigma]$, we thus have with probability at least $1-\delta$,

$$\widehat{\sigma} \in \text{argmin}_{\sigma \in \mathcal{S}_n} \text{er}^{\text{PD}}_{\mu,\mathbf{P}}[\sigma] \,.$$

This proves the result. □

Proof of Proposition 19

Proof. Suppose $\mathbf{P}$ satisfies the GLN condition with vector $\boldsymbol{\alpha} \in \mathbb{R}^n$. Then clearly, since $P_{ij} \neq \frac{1}{2} \; \forall i \neq j$, we have $\forall i < j$:

$$z_{ij} = 1 \implies P_{ji} > P_{ij} \implies \sum_{k=1}^{n} \alpha_k P_{ki} > \sum_{k=1}^{n} \alpha_k P_{kj} \implies \boldsymbol{\alpha}^\top(\mathbf{P}_i - \mathbf{P}_j) > 0$$

$$z_{ij} = -1 \implies P_{ij} > P_{ji} \implies \sum_{k=1}^{n} \alpha_k P_{kj} > \sum_{k=1}^{n} \alpha_k P_{ki} \implies \boldsymbol{\alpha}^\top(\mathbf{P}_i - \mathbf{P}_j) < 0 \,.$$

Thus $S_{\mathbf{P}}$ is linearly separable by the hyperplane $\boldsymbol{\alpha}$ passing through the origin.

Conversely, suppose that $S_{\mathbf{P}}$ is linearly separable by a hyperplane passing through the origin. Then $\exists \boldsymbol{\alpha} \in \mathbb{R}^n$ s.t. $z_{ij}\boldsymbol{\alpha}^\top(\mathbf{P}_i - \mathbf{P}_j) > 0 \; \forall i < j$. Thus we have $\forall i < j$:

$$P_{ij} > P_{ji} \implies z_{ij} = -1 \implies \boldsymbol{\alpha}^\top(\mathbf{P}_i - \mathbf{P}_j) < 0 \implies \sum_{k=1}^{n} \alpha_k P_{kj} > \sum_{k=1}^{n} \alpha_k P_{ki}$$

$$P_{ji} > P_{ij} \implies z_{ij} = 1 \implies \boldsymbol{\alpha}^\top(\mathbf{P}_i - \mathbf{P}_j) > 0 \implies \sum_{k=1}^{n} \alpha_k P_{ki} > \sum_{k=1}^{n} \alpha_k P_{kj} \,.$$

Thus $\mathbf{P}$ satisfies the GLN condition. □

Proof of Theorem 20

Proof. Let m satisfy the given conditions. We first show that with probability at least $1 - \frac{\delta}{2}$, every label $\text{sign}(\widehat{P}_{ji} - \widehat{P}_{ij})$ in $S_{\widehat{\mathbf{P}}}$ is the same as the corresponding label $\text{sign}(P_{ji} - P_{ij})$ in $S_{\mathbf{P}}$. We have,

$$\begin{aligned} \mathbf{P}\Big(\exists i \neq j : \big|\widehat{P}_{ij} - P_{ij}\big| \geq \gamma\Big) &\leq \sum_{i \neq j} \mathbf{P}\Big(\big|\widehat{P}_{ij} - P_{ij}\big| \geq \gamma\Big) \,, \quad \text{by union bound} \\ &\leq 4n^2 \exp\left(\frac{-m\gamma^2\mu^2_{\min}}{128}\right) , \quad \text{by Lemma 3 (part 4)} \\ &\qquad \text{(since } m \geq \tfrac{128}{\gamma^2\mu^2_{\min}} \log(\tfrac{8n^2}{\delta}) \geq \tfrac{1}{\mu_{\min}} \ln(\tfrac{2}{\gamma})) \\ &\leq \frac{\delta}{2} \,, \quad \text{since } m \geq \tfrac{128}{\gamma^2\mu^2_{\min}} \log(\tfrac{8n^2}{\delta}). \end{aligned}$$

Thus we have that with probability at least $1-\frac{\delta}{2}$,

$$|\widehat{P}_{ij} - P_{ij}| \le \gamma \quad \forall i \neq j\,.$$

By definition of γ, this yields that with probability at least $1-\frac{\delta}{2}$,

$$\widehat{P}_{ij} > \widehat{P}_{ji} \Longleftrightarrow P_{ij} > P_{ji} \quad \forall i \neq j\,,$$

i.e. with probability at least $1-\frac{\delta}{2}$,

$$\text{sign}(\widehat{P}_{ji} - \widehat{P}_{ij}) = \text{sign}(P_{ji} - P_{ij}) \quad \forall i < j\,.$$

Next, we show that with probability at least $1-\frac{\delta}{2}$, every point $(\widehat{\mathbf{P}}_i - \widehat{\mathbf{P}}_j)$ in $S_{\widehat{\mathbf{P}}}$ falls on the same side of the hyperplane given by $\boldsymbol{\alpha}$ as the corresponding point $(\mathbf{P}_i - \mathbf{P}_j)$ in $S_{\mathbf{P}}$. We have,

$$\begin{aligned}
&\mathbf{P}\Big(\exists (i<j) : \|(\widehat{\mathbf{P}}_i - \widehat{\mathbf{P}}_j) - (\mathbf{P}_i - \mathbf{P}_j)\|_2 \;\ge\; \frac{r^{\boldsymbol{\alpha}}_{\min}}{2}\Big) \\
&= \mathbf{P}\Big(\exists (i<j) : \|(\widehat{\mathbf{P}}_i - \mathbf{P}_i) - (\widehat{\mathbf{P}}_j - \mathbf{P}_j)\|_2 \;\ge\; \frac{r^{\boldsymbol{\alpha}}_{\min}}{2}\Big) \\
&\le \sum_{i<j} \mathbf{P}\Big(\|(\widehat{\mathbf{P}}_i - \mathbf{P}_i) - (\widehat{\mathbf{P}}_j - \mathbf{P}_j)\|_2 \ge \frac{r^{\boldsymbol{\alpha}}_{\min}}{2}\Big)\,, \quad \text{by union bound} \\
&\le \sum_{i<j} \bigg(\mathbf{P}\Big(\|\widehat{\mathbf{P}}_i - \mathbf{P}_i\|_2 \ge \frac{r^{\boldsymbol{\alpha}}_{\min}}{4}\Big) + \mathbf{P}\Big(\|\widehat{\mathbf{P}}_j - \mathbf{P}_j\|_2 \ge \frac{r^{\boldsymbol{\alpha}}_{\min}}{4}\Big)\bigg) \\
&\le \sum_{i<j} \bigg(\mathbf{P}\Big(\exists k : |\widehat{P}_{ki} - P_{ki}| \ge \frac{r^{\boldsymbol{\alpha}}_{\min}}{4\sqrt{n}}\Big) + \mathbf{P}\Big(\exists k : |\widehat{P}_{kj} - P_{kj}| \ge \frac{r^{\boldsymbol{\alpha}}_{\min}}{4\sqrt{n}}\Big)\bigg) \\
&\le \sum_{i<j} \bigg(\sum_k \mathbf{P}\Big(|\widehat{P}_{ki} - P_{ki}| \ge \frac{r^{\alpha}_{\min}}{4\sqrt{n}}\Big) + \sum_k \mathbf{P}\Big(|\widehat{P}_{kj} - P_{kj}| \ge \frac{r^{\alpha}_{\min}}{4\sqrt{n}}\Big)\bigg) \\
&\le 8n^3 \exp\left(\frac{-m(r^{\boldsymbol{\alpha}}_{\min})^2 \mu^2_{\min}}{2048n}\right)\,, \quad \text{by Lemma 3 (part 4)} \\
&\qquad (\text{since } m \ge \tfrac{2048\,n}{(r^{\boldsymbol{\alpha}}_{\min})^2 \mu^2_{\min}} \log(\tfrac{16n^3}{\delta}) \ge \tfrac{1}{\mu_{\min}} \ln(\tfrac{8\sqrt{n}}{r^{\boldsymbol{\alpha}}_{\min}})) \\
&\le \frac{\delta}{2}\,, \quad \text{since } m \ge \tfrac{2048\,n}{(r^{\boldsymbol{\alpha}}_{\min})^2 \mu^2_{\min}} \log(\tfrac{16n^3}{\delta}).
\end{aligned}$$

Thus with probability at least $1-\frac{\delta}{2}$,

$$\|(\widehat{\mathbf{P}}_i - \widehat{\mathbf{P}}_j) - (\mathbf{P}_i - \mathbf{P}_j)\|_2 \;\le\; \frac{r^{\boldsymbol{\alpha}}_{\min}}{2} \quad \forall i < j\,.$$

By definition, $r^{\boldsymbol{\alpha}}_{\min}$ is the smallest Euclidean distance of any point $(\mathbf{P}_i - \mathbf{P}_j)$ to the hyperplane defined by $\boldsymbol{\alpha}$; therefore we get that with probability at least $1-\frac{\delta}{2}$, all points $(\widehat{\mathbf{P}}_i - \widehat{\mathbf{P}}_j)$ fall on the same side of the hyperplane $\boldsymbol{\alpha}$ as the corresponding points $(\mathbf{P}_i - \mathbf{P}_j)$.

Combining the above statements yields that with probability at least $1-\delta$, the dataset $S_{\widehat{\mathbf{P}}}$ is also linearly separable by $\boldsymbol{\alpha}$; in this case, the SVM-RankAggregation algorithm produces a vector $\widehat{\boldsymbol{\alpha}}$ that correctly classifies all examples in $S_{\widehat{\mathbf{P}}}$, i.e. satisfies $z_{ij}\widehat{\boldsymbol{\alpha}}^\top(\widehat{\mathbf{P}}_i - \widehat{\mathbf{P}}_j) > 0\; \forall i<j$ (where $z_{ij} = \text{sign}(P_{ji} - P_{ij})$), and it can be verified that $\widehat{\boldsymbol{\alpha}}$ must then also satisfy $z_{ij}\widehat{\boldsymbol{\alpha}}^\top(\mathbf{P}_i - \mathbf{P}_j) > 0\; \forall i<j$, so that $\text{argsort}(\widehat{\boldsymbol{\alpha}}) \subseteq \text{argmin}_{\sigma \in \mathcal{S}_n} \text{er}^{\text{PD}}_{\mu,\mathbf{P}}[\sigma]$. This yields that with probability at least $1-\delta$,

$$\widehat{\sigma} \in \text{argmin}_{\sigma \in \mathcal{S}_n} \text{er}^{\text{PD}}_{\mu,\mathbf{P}}[\sigma]\,.$$

□