

Approximate Linear Time ML Decoding on Tail-Biting Trellises in Two Rounds

K. Murali Krishnan , Priti Shankar
 Dept. of Computer Science and Automation
 Indian Institute of Science, Bangalore - 560012, India
 Email: {kmurali,priti}@csa.iisc.ernet.in

Abstract—A linear time approximate maximum likelihood decoding algorithm on tail-biting trellises is presented, that requires exactly two rounds on the trellis. This is an adaptation of an algorithm proposed earlier with the advantage that it reduces the time complexity from $O(m \log m)$ to $O(m)$ where m is the number of nodes in the tail-biting trellis. A necessary condition for the output of the algorithm to differ from the output of the ideal ML decoder is deduced and simulation results on an AWGN channel using tail-biting trellises for two rate 1/2 convolutional codes with memory 4 and 6 respectively, are reported.

I. INTRODUCTION

Maximum likelihood decoding on tail-biting trellises (TBT) has been extensively studied in the literature and several linear time approximate algorithms have been proposed, (see for example, [7], [6], [10], [9], [3]). Some of these algorithms may fail to converge on certain inputs. Algorithms with guaranteed convergence were studied in [11], but they fail to achieve linear complexity. In particular, although the approximate algorithm proposed in [11], achieves performance close to an ideal ML decoder, it has a worst case time complexity of $O(m \log m)$, where m is the number of nodes in the TBT. The algorithm exploits the fact that a linear tail-biting trellis can be viewed as a coset decomposition of the group corresponding to the linear code with respect to a specific subgroup and is an adaptation of the classical A^* algorithm. The algorithm operates in two phases. The first phase does a Viterbi-like pass on the TBT to obtain certain *estimates* which are used in the second phase to guide the search for the shortest path corresponding to a codeword in the TBT.

In this note, the complexity of the approximate algorithm in [11] is reduced to $O(m)$. The reduction in complexity is achieved by eliminating the use of a heap in the second phase of the original algorithm using the well known technique of dynamic programming. The estimates gathered during the first phase are used in the second phase for the computation of a *metric* for each node in the TBT using another simple Viterbi-like pass. It turns out that updates performed by the two algorithms are identical for the shortest path which must be output by the algorithm (although the metric values computed for other nodes may differ).

We give an analysis of the algorithm here. Simulations are included for completeness and the two algorithms perform identically as expected.

II. BACKGROUND

A linear tail-biting trellis for an (n, k) linear block code C over field F_q can be constructed as a *trellis product* [5] of the representation of the individual trellises corresponding to the k rows of the generator matrix G for C [4]. The trellis product T of a pair of trellises T_1 and T_2 will have at $Timeindex(i)$ a set of vertices which is the Cartesian product of vertices of T_1 and T_2 at that time index, with an edge between $Timeindex(i)$ and $Timeindex(i+1)$ from (v_1, v_2) to (v'_1, v'_2) , with label $(a + a')$ whenever (v_1, v'_1) and (v_2, v'_2) are edges between vertices at $Timeindex(i)$ and $Timeindex(i+1)$ in T_1 and T_2 with labels a and a' respectively for some $a, a' \in F_q$, $0 \leq i \leq n-1$, where $+$ denotes addition in F_q . Let \vec{g}_i , $1 \leq i \leq k$ be the rows of a generator matrix G for the linear code C . Each vector \vec{g}_i generates a one-dimensional subcode of C , which we denote by C_i . Therefore $C = C_1 + C_2 + \dots + C_k$, and the trellis representing C is given by $T = T_1 \times T_2 \times \dots \times T_k$, where T_i is the trellis for \vec{g}_i , $1 \leq i \leq k$. Given a codeword $\vec{c} = \langle c_1, c_2, \dots, c_n \rangle \in C$, the *linear span* [5] of \vec{c} , is the interval $[i, j] \in \{1, 2, \dots, n\}$ which contains all the non-zero positions of \vec{c} . A *circular span* [4] has exactly the same definition with $i > j$. Note that for a given vector, the linear span is unique, but circular spans are not. For a vector $\vec{x} = \langle x_1, \dots, x_n \rangle$ over the field F_q , there is a unique *elementary trellis* [5], [4] representing \vec{x} [4]. This trellis has q vertices at time indices i to $(j-1) \bmod n$, and a single vertex at other positions. Consequently, T_i in the trellis product mentioned earlier, is the elementary trellis representing \vec{g}_i for some choice of span (either linear or circular). Koetter and Vardy [4] have shown that any linear trellis, conventional or tail-biting can be constructed from a generator matrix whose rows can be partitioned into two sets, those which have linear span, and those taken to have circular span. The trellis for the code is formed as a product of the elementary trellises corresponding to these rows. We will represent such a generator matrix as $G = \begin{bmatrix} G_l \\ G_c \end{bmatrix}$, where G_l is the submatrix consisting of rows with linear span, and G_c the submatrix of rows with circular span. Let T_l denote the minimum conventional trellis for the code generated by G_l . If l is the number of rows of G with linear span and c the number of rows of circular span, the tail-biting trellis constructed using the product construction will have q^c start states. where, each such start state defines

a subtrellis whose codewords form a coset of the subcode corresponding to the subtrellis containing the all 0 codeword.

For the description of the decoding algorithm we assume a tail-biting trellis with start states $s_0, s_1 \dots s_t$ and final states $f_0, f_1 \dots f_t$. where t is the number of subtrellises. An (s_i, f_i) path is a codeword path in trellis T_i , whereas an (s_i, f_j) path for $i \neq j$ is a non codeword path. For purposes of our discussion we term the edge label sequence along such a path as a *semi-codeword* as in [11]. We assume an AWGN channel with binary antipodal signalling. When the edges are given weights corresponding to the log-likelihood values, ML decoding corresponds to finding the minimum weight codeword path in the TBT.

III. THE TWO PHASE ALGORITHM

The algorithm operates in two phases, each taking linear time. The first phase is a Viterbi pass which computes a function $Cost()$ for each vertex u in the trellis. This value of cost is used by the second phase to compute a *metric* at each vertex of the trellis. The final decoding decision will be based on the *metric* values at the final nodes of the trellis.

Let $l(u, v)$ denote the length of the shortest path connecting vertices u and v in the tail-biting trellis. Note that $l()$ satisfies the *triangular inequality*. i.e., $l(u, v) \leq l(u, w) + l(w, v)$ for all nodes u, v, w in the trellis. A *codeword* is an $s_i - f_i$ path while a *semi-codeword* is an $s_j - f_i$ path, $i, j \in \{1, \dots, t\}$, where t is the number of subtrellises. Note that all codewords are semi-codewords. Define $\delta(u) = \min_{1 \leq i \leq t} l(s_i, u)$. We say an edge $(u, v) \in Section(i)$ if $v \in TimeIndex(i)$. Define the *metric at node u for trellis i* $m_i(u) = l(s_i, u) + \delta(f_i) - \delta(u)$. Define *metric at node u* , $m(u) = \min_{1 \leq i \leq t} m_i(u)$.

Suppose $\delta(u) = x$ and this is the length of an $s_i - u$ path, the first phase of the algorithm assigns the program variable $Cost[u]$ the value x and $SurvTrellis[u]$ the value i . We call the the $s_i - u$ path corresponding to this assignment the *survivor at u* .

These values are used to assign values to the program variable $Metric[u]$ in the second phase, which is intended to store the value of the metric $m(u)$. The trellis corresponding to the minimum metric value is stored in the variable $Trellis[u]$. However, the values assigned to $Metric[]$ can be incorrect, in that it is not equal to $m()$. The algorithm may even fail to assign a value to $Metric[u]$ for every node u . We shall derive the conditions under which the algorithm may fail to decode correctly.

The program variable $Dist[]$ stores the length of the path to the node corresponding to the minimum value of $Metric[]$ in the second phase. The program variable $Pred[]$ used in both the phases stores the predecessor along the paths traced to the node by the algorithm in the respective phases.

The function $Member((u, v), i)$ assumed in the algorithm description below takes as input an edge (u, v) and integer i and returns TRUE if the edge (u, v) belongs to trellis T_i , FALSE otherwise. Note that the function $Member()$ needs only $O(1)$ lookup time although the lookup table is of size quadratic on the number of vertices in the trellis.

A. Phase 1: Estimation

Initialization:

```
for each  $s_i \in TimeIndex(0)$ 
   $Cost[s_i] = 0$ 
   $SurvTrellis[s_i] = i$ 
   $Pred[s_i] = s_i$ 
for each  $v \notin TimeIndex(0)$   $cost[v] = \infty$ 
```

Estimation:

```
for  $Timeindex := 1$  to  $n$  do
  for each edge  $(u, v) \in Section(i)$  do
     $Temp = Cost[u] + l[u, v]$ 
    if  $(Cost[v] > Temp)$  then
       $Cost[v] = Temp$ 
       $Pred[v] = u$ 
       $SurvTrellis[v] = SurvTrellis[u]$ 
```

Clearly by the end of this phase, $Cost[u] = \delta(u)$ for each vertex u in the trellis.

Let $j = \operatorname{argmin}_{1 \leq i \leq t} \delta(f_i)$. If the algorithm assigns $SurvTrellis[f_j] = j$, then *survivor at f_j* which corresponds to the minimum weight semi-codeword in the trellis turns out to be a codeword and the algorithm stops. Otherwise, the second phase described below will be executed.

B. Phase 2: Revision

Initialization:

```
for each  $s_i \in TimeIndex(0)$ 
  if  $(Survivor[f_i] \neq i)$  then  $Metric[s_i] = \delta(f_i)$ 
  else  $Metric[s_i] = \infty$  /* No processing for  $T_i$  */
   $Pred[s_i] = s_i$ 
   $Trellis[s_i] = i$ 
  if  $(Metric[s_i] = \infty)$  then  $Dist[s_i] = \infty$ 
  else  $Dist[s_i] = 0$ 
for each  $v \notin TimeIndex(0)$   $Metric[v] = \infty$ 
```

Revision

```
for  $Timeindex := 1$  to  $n$  do
  for each edge  $(u, v) \in Section(i)$  do
     $Update(u, v)$ 
 $Update(u, v)$ 
  if  $(\text{notMember}((u, v), Trellis[u]))$  return;
   $temp = Dist[u] + l[u, v] + Cost[f_{Trellis[u]}] - Cost[v]$ 
  if  $(Metric[v] > temp)$  then
     $Metric[v] = temp$ 
     $Pred[v] = u$ 
     $Trellis[v] = Trellis[u]$ 
     $Dist[v] = Dist[u] + l[u, v]$ 
```

The second phase attempts to compute the value of the *metric*, $m(u)$ for each vertex u of the trellis. If the first phase assigned $SurvTrellis[f_i] = i$ for some final node f_i , for

the particular trellis T_i the second phase processing is not required. We say a Trellis T_i participates in the second phase if $\text{SurvTrellis}[f_i] \neq i$ and $\delta(f_i) \leq \min_{j, \text{SurvTrellis}[f_j]=j} \delta(f_j)$. The final decoding decision is based on the values of the metric at the final nodes of the trellis. We shall derive the conditions under which the algorithm will achieve maximum likelihood decoding on a tail-biting trellis for a linear code, when binary antipodal signalling is used over an AWGN channel.

C. Final Decision

If the algorithm does not stop in the first phase, choose vertex $j = \text{argmin}_{1 \leq i \leq t} \text{Metric}[f_i]$. The output of the algorithm is the codeword corresponding to the $s_j - f_j$ path obtained by tracing the predecessors of f_j till s_j . The array $\text{Pred}()$ stores the predecessors of each node along the path that minimizes the value of metric . Note that if T_j does not participate in the second phase, the path must be traced along $\text{Pred}()$ values in the first phase.

IV. ANALYSIS

For any node u , if $\text{Trellis}[u] = j$, then $\text{Dist}[u] \geq l(s_j, u)$ because the value assigned $\text{Dist}[u]$ is the length of an $s_j - u$ path. Consequently $\text{Metric}[u] \geq m_j(u)$. We collect these facts into a lemma:

Lemma 1: During the second phase, if the algorithm assigns for a node u , $\text{Trellis}[u] = j$ then $\text{Dist}[u] \geq l(s_j, u)$ and $\text{Metric}[u] \geq m_j(u)$.

The following simple property of $\delta()$ will be useful:

Lemma 2: If (u, v) is an edge in the TBT, the $\delta(v) \leq \delta(u) + l(u, v)$.

Proof: The shortest path from a start node to v cannot be longer than the shortest path from a start node to v through u . ■

The following lemma asserts that the value assigned to Metric by the algorithm cannot be smaller than the Metric value of its predecessor node.

Lemma 3: Let (u, v) be an edge in the Tail-biting Trellis. Let $\text{Trellis}[u] = i$. Suppose the second phase assigns $\text{Pred}[v] = u$ then $\text{Metric}[v] \geq \text{Metric}[u]$

Proof: An inspection of the algorithm reveals that the algorithm assigns to $\text{Dist}[u]$ the cost of some $s_i - u$ path. Hence $\text{Dist}[u] \geq l(s_i, u)$. By the Metric update rule of the algorithm, $\text{Metric}[v] = \text{Dist}[u] + l(u, v) + \delta(f_i) - \delta(u)$. Since $\text{Metric}[u] = \text{Dist}[u] + \delta(f_i) - \delta(u)$, the result follows as $\delta(v) \leq \delta(u) + l(u, v)$ by lemma 2. ■

Corollary 1: If the algorithm assigns $\text{Trellis}[u] = i$, then $\text{Metric}[u] \geq \text{Metric}[s_i] = \delta(f_i)$

Proof: The algorithm initializes $\text{Metric}[s_i]$ to $\delta(f_i)$. By previous lemma, the value cannot decrease along any $s_i - u$ path. ■

The algorithm, if assigns any value, must set $\text{Trellis}[f_j] = j$ and $\text{Metric}[f_j] = \text{Dist}[f_j] \geq l(s_j, f_j) = m_j(f_j)$ for each $j \in \{1, \dots, t\}$. Thus, if the shortest path corresponding to a codeword in the trellis is an $s_j - f_j$ path, then if $\text{Metric}[f_j] = l(s_j, f_j)$ the algorithm is guaranteed to decode correctly. In the

following, we derive a condition necessary for the algorithm to fail.

Theorem 1: If the shortest codeword corresponds to an $s_i - f_i$ path P , and if P corresponds to the codeword output by a maximum likelihood decoder, then, the two phase algorithm fails to assign $\text{Metric}[u] = m_i(u)$ and $\text{Trellis}[u] = i$ for any node u in P only if there exists $k \neq j \neq i$ such that $l(s_k, f_j) \leq l(s_i, f_i)$.

Proof: Without loss of generality, assume that the all zero codeword was transmitted and an ideal ML decoder will output the all zero codeword. Again, without loss of generality let $P = (s_1 =)u_0, u_1 \dots u_{n-1}, u_n (= f_1)$ be the shortest $s_i - f_i$ path in the sub-trellis T_1 corresponding to the all zero codeword. We therefore have $l(s_1, f_1) < l(s_i, f_i)$ for all $1 < i \leq t$. Let u , be the first node along the path P where there exists some $1 < j \leq t$ such that $m_j(u) \leq m_1(u)$. such node u must exist for otherwise, the algorithm will decode correctly as it will assign $\text{Trellis}[u_i] = 1$ with $\text{Metric}[u_i] = m_1(u_i)$ all along the path P .

Note that $m_1(f_1) = l(s_1, f_1)$ is the value the algorithm would have assigned to $\text{Metric}[f_1]$ if the algorithm had assigned $\text{Trellis}[u_i] = 1$ all along the path P . As the algorithm assigns the minimum value of Metric possible for each node, by lemma 3, it must be true that the actual value assigned to the $\text{Metric}[u]$ by the algorithm must satisfy $\text{Metric}[u] \leq l(s_1, f_1)$. Since we assume that the algorithm assigned $\text{Trellis}[u] = j$, the value of the metric computed at u must have followed an $s_j - u$ path and consequently $\text{Metric}[u] \geq \text{Metric}[s_j] = \delta(f_j)$ (Corollary 1). Hence $\delta(f_j) \leq l(s_1, f_1)$.

Now, Assume that the survivor at f_j is an $s_k - f_j$ path, if $k = j$, we have $l(s_j, f_j) \leq l(s_1, f_1)$, a contradiction. Otherwise, the condition stated in theorem holds. ■

Now to specialize the above to AWGN channel with binary antipodal signalling. The following two results proved in [11] are repeated here for completeness.

Lemma 4: The space of semi-codewords is a vector space.

Proof: Assume that each of the c vectors in the submatrix G_c of the generator matrix is of the form $v_i = [\vec{h}_i, \vec{0}, \vec{t}_i]$ where \vec{h}_i stands for the sequence of symbols before the zero run, and is called the *head* and \vec{t}_i stands for the sequence of symbols following the zero run and is called the *tail* and $\vec{0}$ is the zero run containing the appropriate number of zeroes. Let $\{v_1, v_2 \dots v_c\}$ be the vectors of G_c . Then the matrix G_s defined as $G_s = \begin{bmatrix} G_l \\ G'_c \end{bmatrix}$, where G'_c consists of $2c$ rows of the form $[\vec{h}_i, \vec{0}], [\vec{0}, \vec{t}_i], 1 \leq i \leq c$, generates the set of labels of all paths from any start node to any final node. ■

The following is due to Tendolkar and Hartmann [8].

Lemma 5: Let H be the parity check matrix of the code and let a codeword \vec{x} be transmitted as a signal vector $s(\vec{x})$. Let the binary quantization of the received vector $\vec{r} = r_1, r_2, \dots, r_n$ be denoted by \vec{y} . Let $\vec{r}^T = (|r_1|, |r_2|, \dots, |r_n|)$ and $S = \vec{y}H^T$. Then maximum likelihood decoding is achieved by decoding a received vector \vec{r} into the codeword $\vec{y} + \vec{e}$ where \vec{e} is a binary

vector that satisfies $S = \vec{e}H^T$ and has the property that if \vec{e}' is any other binary vector such that $S = \vec{e}'H^T$ then $\vec{e}.r' < \vec{e}'.r'$ where $.$ is the inner product.

Combining all the above, we have the following necessary condition for error.

Theorem 2: Assume the $\vec{0}$ codeword is the ML codeword corresponding to the path $s_1 - f_1$ in the tail biting trellis. Let \vec{y} be the binary quantization of the received vector. Let r, r' be as defined in Lemma 5. For the error pattern \vec{e} the two phase algorithm decodes to a vector to a vector $\vec{\alpha} \neq \vec{0}$ correspond an $s_j - f_j$ path $j \neq i$ only if there exists a semi-codeword C_s satisfying

$$(C_s + \vec{e}).r' \leq \vec{e}.r' < (C + \vec{e}).r'$$

for all nonzero codewords C , where the semi-codeword c_s either shares either its head or tail with Trellis j .

Proof: Since the ideal ML decoder decodes \vec{y} to $\vec{0}$, we have $\vec{y} + \vec{e} = 0$ or $e = y$. Let H be the parity check matrix of the code while H_s the parity check matrix for the semi-codeword vector space established in Lemma 3. Any binary error vector \vec{e}' which gives the same syndrome as e must belong to the same coset of the code and hence must have the form $C + \vec{e}$, where C is a codeword. Applying Lemma 5, we get $\vec{e}.r' < (C + \vec{e}).r'$ for all codewords C , which proves the right inequality.

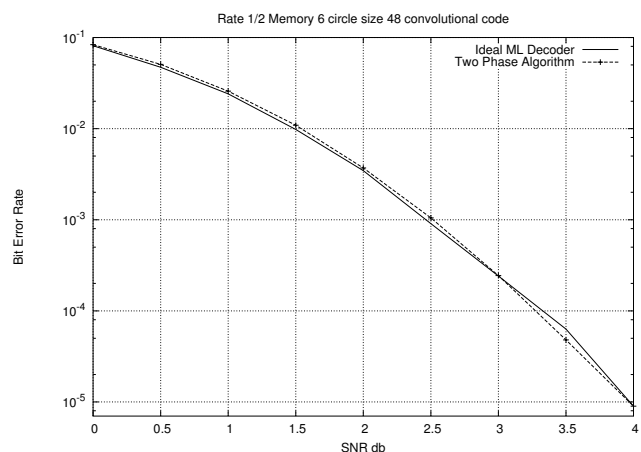
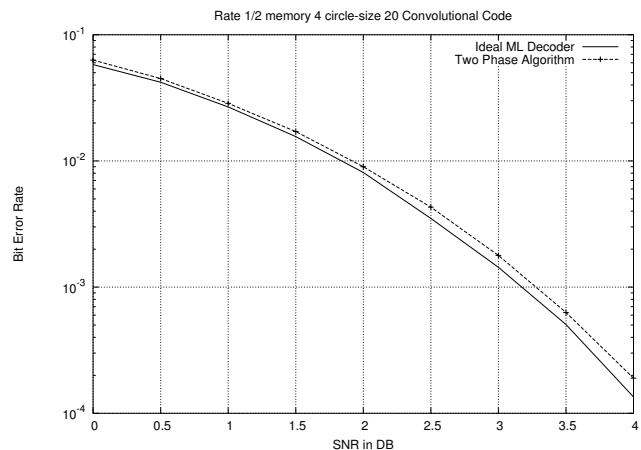
To yield the left inequality, first observe that the first phase of the algorithm does an ML decoding on the semi-codeword space. Any $s_k - f_j$ path P in the tail-biting trellis with $k \neq j$ and $l(s_k, f_j) \leq l(s_1, f_1)$ corresponds to a semi-codeword that an ideal ML decoder operating on the space of semi-codewords will prefer to the all zero codeword. Hence, by applying Lemma 5 to this case and arguing identically as above, we find that for each path such P there must exist a semi-codeword C_s such that $(C_s + \vec{e}).r' \leq \vec{e}.r'$. The claim follows as Theorem 1 asserts that this condition is necessary for the algorithm to fail to decode the received vector to the all zero codeword. ■

V. COMPLEXITY

Since each phase takes linear time, the algorithm runs in time linear in the size of the tail-biting trellis. As each pass is Viterbi like, the worst case number of comparisons performed is bounded by twice that of the Viterbi algorithm. The space complexity is quadratic in the size of the trellis owing to the lookup table of size $t|V|$ required for the *member()* function, where $|V|$ is the number of vertices in the trellis. However, this is not a serious drawback as the table can be efficiently implemented using bit vector representation.

VI. SIMULATION RESULTS

The results of simulations on an AWGN channel for the two phase algorithm are displayed in the figures below. The codes used are a rate 1/2 memory 6 convolutional code with a circle size of 48 (same as the (554,744) code convolutional code used in [2]) and a rate 1/2 memory 4 convolutional code with circle size 20 (same as the (72,62) code used in [1]). The performance of the above codes is compared with that of the



exact ML decoding algorithm in [11]. It is seen that the bit error rate of the algorithm approaches that of the ideal ML decoder.

VII. DISCUSSION AND CONCLUSION

The performance of the algorithm can be improved at the expense of more storage by tracking more than one paths corresponding lowest values of *Metric* during the second phase. However, the time complexity increases proportional to the number of stored paths. Practice has shown that memorizing the best two paths corresponding to the minimum value of *Metric* at each node gives performance almost indistinguishable from the ideal maximum likelihood decoder [11]

An interesting failure condition of the algorithm is the following: The algorithm may fail to assign a value to the *Metric* field for a node if in the second phase a node fail to belong to any of the trellises assigned to the *Trellis* field of its predecessors by the algorithm. If this happens along all paths to all final states, the algorithm may fail to output a codeword in the second phase. Note that the error condition

proved handles this case as well. However this situation never occurred in simulations performed.

From the results of simulations on the rate 1/2, memory 4 convolutional code with a circle size of 20 and a rate 1/2 memory 6 convolutional code with a circle size of 48, it is seen that the algorithm performs close to the ideal ML decoder. The performance is comparable with other linear time approximate methods. The present algorithm reduces computation to just two Viterbi computation on the tail-biting trellis and does not require dynamic data structures like the heap necessary in the original versions using the A* algorithm [11].

VIII. ACKNOWLEDGMENT

The authors gratefully acknowledge the help of Madhu A. S. in implementing the algorithm and helping with the simulations.

REFERENCES

- [1] J.B. Anderson and S.M. Hladik, An Optimal Circular Viterbi Decoder for the Bounded Distance Criterion, *IEEE Transactions on Communications*, **50**(11), November 2002.
- [2] J.B. Anderson and S.M. Hladik, Tail-biting MAP decoders, *IEEE J. in Selected Areas of Communication*, **16**(2), Feb. 1998.
- [3] R.V. Cox and C.V. Sundberg, An Efficient Adaptive Circular Viterbi Algorithm for Decoding Generalized Tailbiting Convolutional Codes, *IEEE Transactions on Vehicular Technology* **43**(1), February 1994, pp 57-68.
- [4] R. Koetter and A. Vardy, The Structure of Tail-Biting Trellises: Minimality and Basic Principles, *IEEE Trans. Inform. Theory*, September 2003, pp. 2081-2105.
- [5] F.R. Kschischang and V. Sorokine, On the trellis structure of block codes, *IEEE Trans. Inform. Theory*, **41**(6), Nov 1995, pp. 1924-1937.
- [6] J.H.Ma and J.K.Wolf, On tail-biting convolutional codes, *IEEE Trans. Commun.*, **34**, February 1986, pp 104-111.
- [7] R.Y.Shao, Shu Lin and M.P.C. Fossorier, Two decoding algorithms for tail-biting codes, *IEEE Trans. Commun.*, **51**(10), October 2003, pp 1658-1665.
- [8] N.N. Tendolkar and C.R.P. Hartmann, Generalization of Chase Algorithms for Soft Decision Decoding of Binary Linear Codes, *IEEE Trans. Inform.Theory*, **30**(5), September 1984, pp 714-721.
- [9] Q.Wang and V.K.Bhargava, An efficient maximum-likelihood decoding algorithm for generalized tail-biting codes including quasi-cyclic codes, *IEEE Trans. Commun.*, **37**, August 1989, pp 875-879.
- [10] K.S.Zigangirov and V.V. Chepyzhov, Study of decoding tail-biting convolutional codes, in *Proc. Swedish-Soviet Workshop on Information Theory*(Gotland, Sweden, Aug. 1989), pp 52-55.
- [11] Priti Shankar, P.N.A. Kumar, K. Sasidharan, B.S. Rajan, A.S. Madhu, Efficient Convergent Maximum Likelihood Decoding on Tail-Biting Trellises, available at <http://arxiv.org/abs/cs.IT/0601023>