DAA 2007: Mid-Term Exam I

Q1.[4] Give an efficient push-relabel algorithm to find a maximum matching in a bipartite graph. Analyse your algorithm.

Q2.[4] Give an efficient incremental algorithm for solving a 3-dimensional linear program. That is, you are given n linear constraints:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &\leq b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 &\leq b_n \end{aligned}$$

You have to find the point (x_1, x_2, x_3) satisfying the above linear constraints which maximizes the objective function $c_1x_1 + c_2x_2 + c_3x_3$. (You can assume that the first 3 constraints of the above LP bound the problem in the direction of increasing objective function.)

Q3.[4] In an undirected unweighted graph $G = (V, E)$, the Max-Cut problem is a partition $(A, V - A)$ which maximizes the number of edges with one endpoint in A and the other in $V - A$. Unlike the min-cut problem, the max-cut problem is hard to solve. Consider the following randomized algorithm to get an approximate solution for this problem:

1. $A = \emptyset; B = \emptyset$.
2. for every vertex $v \in V$ do:

   toss a fair coin - if the outcome is heads, put v in A else put v in B.
3. Return the cut (A, B) as a candidate max-cut.

Show that the expected number of edges between A and B is at least half the size of the max-cut.

Q4.[4] Show that the number of distinct minimum cuts (here we are talking of the *global* min-cut) in an undirected graph (assume all edge weights are one) is at most $\binom{n}{2}$. That is, show that the number of distinct cuts whose value is equal to the value of the min cut in the graph is at most $\frac{n(n-1)}{2}$.

Q5.[4] Let $G = (V, E)$ be an undirected graph with non-negative weights on its edges. For any $s, t \in V$, let $f(s, t)$ denote the weight of a minimum s-t cut in G. Let u, v, w be any 3 vertices in V. Suppose $f(u, v) \leq f(u, w) \leq f(v, w)$. Then show that $f(u, v) = f(u, w)$, i.e., the smaller two numbers have to be equal.