

## DAA 2007: Mid-Term Exam I

**Q1.[4]** Give an efficient push-relabel algorithm to find a maximum matching in a bipartite graph. Analyse your algorithm.

**Q2.[4]** Give an efficient incremental algorithm for solving a 3-dimensional linear program. That is, you are given  $n$  linear constraints:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &\leq b_2 \\ &\cdot \\ &\cdot \\ &\cdot \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 &\leq b_n \end{aligned}$$

You have to find the point  $(x_1, x_2, x_3)$  satisfying the above linear constraints which maximizes the objective function  $c_1x_1 + c_2x_2 + c_3x_3$ . (You can assume that the first 3 constraints of the above LP bound the problem in the direction of increasing objective function.)

**Q3.[4]** In an undirected unweighted graph  $G = (V, E)$ , the Max-Cut problem is a partition  $(A, V - A)$  which maximizes the number of edges with one endpoint in  $A$  and the other in  $V - A$ . Unlike the min-cut problem, the max-cut problem is hard to solve. Consider the following randomized algorithm to get an approximate solution for this problem:

1.  $A = \emptyset; B = \emptyset$ .
2. for every vertex  $v \in V$  do:  
toss a fair coin - if the outcome is heads, put  $v$  in  $A$  else put  $v$  in  $B$ .
3. Return the cut  $(A, B)$  as a candidate max-cut.

Show that the expected number of edges between  $A$  and  $B$  is at least half the size of the max-cut.

**Q4.[4]** Show that the number of distinct minimum cuts (here we are talking of the *global* min-cut) in an undirected graph (assume all edge weights are one) is at most  $\binom{n}{2}$ . That is, show that the number of distinct cuts whose value is equal to the value of the min cut in the graph is at most  $\frac{n(n-1)}{2}$ .

**Q5.[4]** Let  $G = (V, E)$  be an undirected graph with non-negative weights on its edges. For any  $s, t \in V$ , let  $f(s, t)$  denote the weight of a minimum  $s$ - $t$  cut in  $G$ . Let  $u, v, w$  be any 3 vertices in  $V$ . Suppose  $f(u, v) \leq f(u, w) \leq f(v, w)$ . Then show that  $f(u, v) = f(u, w)$ , i.e., the smaller two numbers have to be equal.