Program Analysis and Verification

Assignment 3

(Due on Wed 19 Sept 2007)

1. Give a simple example of a lattice with a least and greatest element which is *not* a complete lattice.

2. Let $(D, \leq)$ be a complete lattice.

   (a) Let $f : D \to D$ be a monotonic function. Show that $(D \times D, \sqsubseteq)$ is also a complete lattice, and $f' : (D \times D) \to (D \times D)$ given by $f(x, y) = (f(x), f(y))$, is also monotonic wrt $(D \times D, \sqsubseteq)$.

   (b) Let $[D \to D]$ denote the set of all functions from D to D. Define a natural partial order $\sqsubseteq$ on $[D \to D]$ so that $([D \to D], \sqsubseteq)$ is a complete lattice.

3. Prove the claim made in class that the "Meet over all paths" definition of the vector of reaching definitions is in fact the least fixpoint of the λ function we defined, for the factorial program considered. Show first that the λ function is in fact monotonic wrt the appropriate lattice.

4. Prove that a partial order in which every subset has a *lub*, is also a complete lattice (i.e. every subset also has a *glb*).

5. Let $(D, \leq)$ be a complete lattice. Consider the conditions on $(D, \leq)$ below:

   - (*Ascending Chain Condition*) There are no infinite ascending chains in $(D, \leq)$.
   - (*Continuity*) We say a subset X of D is *directed* if every finite subset of X has an upper bound in X. We then say that $f : D \to D$ is *continuous* (wrt $(D, \leq)$) if for every directed subset X of D we have $f(\bigsqcup X) = \bigsqcup(f(X))$.

   Show that

   (a) If f satisfies continuity then f is monotonic.

   (b) If either f is continuous, or $(D, \leq)$ satisfies Ascending Chain Condition and f is monotonic, then

   $$\mathit{lfp}(f) = \bigsqcup(\{f^n(\bot)\}_{n \in \{0,1\ldots\}}).$$