Eliminating past operators in Metric Temporal Logic

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Abstract

We consider variants of Metric Temporal Logic (MTL) with the past operators S and S_I. We show that these operators can be "eliminated" in the sense that for any formula in these logics containing the S and S_I modalities, we can give a formula over an extended set of propositions which does not contain these past operators, and which is equivalent to the original formula modulo a projection onto the original set of propositions. These results have implications with regard to decidability and closure under projection for some well-known real-time logics in the literature.

1 Introduction

Metric Temporal Logic (MTL) introduced by Koymans [11] is a popular logic for specifying quantitative timing properties of real-time systems. It is interpreted over timed behaviours and extends the "until" modality U of LTL [13] with an interval index, allowing formulas of the form $\varphi U_I \psi$ which say that ψ is satisfied in the future at a time point whose distance lies in the interval I, and φ is held on to at all time points in between. The modalities S and S_I express the symmetric properties in the past. What is considered a possible timepoint to assert a formula depends on whether one considers the "pointwise" or "continuous" interpretation of the logic. In the pointwise semantics we are allowed to make assertions only at time points corresponding to event occurrences, while in the continuous semantics we are allowed to make assertions at arbitrary timepoints. In general, the continuous semantics is strictly more expressive than the pointwise semantics [2, 14].

In this paper we show some results about the equivalence of various fragments of MTL in terms of satisfiability-preserving translations. As a basic stepping stone we first show that the formulas of MTL can be "flattened" in the sense that for any formula in the logic (possibly with the past modality S_I) we can construct a satisfiability-equivalent formula which has no occurrences of nested U_I, S_I, or even S formulas. In fact, the only subformulas involving the above modalities are of the form pU_Iq, pS_Iq, or pSq, where p and q are propositions. We call this the "flat" or "non-recursive" version of MTL. The idea we use is quite simple: to flatten U_I formulas for example, we introduce new propositions p_0 and p_1 for each subformula of the form $\varphi U_I \psi$, replace each occurrence of $\varphi U_I \psi$ by $p_0 U_I p_1$, and add formulas which ensure that p_0 and p_1 correctly capture the truth of φ and ψ along the model. As a simple illustrative example, the flattened form of the formula $(pUq)S_{(0,1)}(p \wedge (qUr))$ is $(p_0 S_{(0,1)} p_1) \wedge \Box(p_0 \Leftrightarrow (pUq)) \wedge \Box(p_1 \Leftrightarrow (p \wedge (qUr)))$ (here $\Box\varphi$ stands for "always φ" or $\neg(\top U \neg\varphi)$). This result is shown for both the pointwise and continuous versions of the logic. To point out a simple consequence of this result, we recall that the pointwise version of MTL over infinite models was shown to be undecidable [12] via a reduction from channel systems to the general (recursive) version of MTL. From our result above it now follows that the corresponding non-recursive fragment of the logic is also undecidable.

Many real-time logics have classical temporal logic (in both the pointwise and continuous semantics) as the base logic to which distance operators are added. We show that for any formula in this base logic extended with the S modality, we can "eliminate" S subformulas from this formula in the sense that we can transform it to a formula over an extended set of propositions, which does not contain any S subformulas, and is equivalent to the original formula modulo a projection to the original set of propositions. This result holds for both the pointwise and continuous semantics. The technique used is to first flatten the S subformulas, then replace each pSq subformula by a new proposition r, and finally add formulas which force r to reflect correctly the truth of pSq along the model. For the pointwise case this last part of the formula is easy to construct. The continuous case is a little less obvious, and one has to consider points of discontinuity for p and q in the model, and ensure that r is updated correctly in the intervals between these points.

Among the implications of the above result is that adding the past modality S to a decidable real-time temporal logic, cannot lead to undecidability. Thus the logics MITL^c (continuous MTL in which only non-singular intervals are allowed) over both finite and infinite words, and MTL^{pw} (pointwise MTL) over finite words, which were shown to be decidable in [1] and [12] respectively, remain decidable even when we add the S modality.

Next, using similar techniques, we show that in continuous time we can eliminate S_I subformulas using U and the distance operator $\Diamond_I$ (which is the same as $\top U_I -$). This gives us the result that adding the S_I modality to a decidable variant of MTL in the continuous semantics, cannot lead to undecidability. A similar result cannot be obtained for the pointwise semantics, since it is known that introducing S_I in pointwise MTL over finite words makes the logic undecidable [6].

Finally, one of our goals was to show that we can eliminate S_I subformulas from the logic $\mathrm{MITL}^c_{S_I}$ in a similar manner. The transformation for eliminating S_I subformulas above does not work here, since it may introduce singular intervals even when there were none in the original formula to begin with. However we show that it is still possible to give a satisfiability-preserving transformation which eliminates S_I subformulas when I is a non-singular interval, without introducing any singular intervals. More precisely, we show that for a given $\mathrm{MITL}^c_{S_I}$ formula we can construct an MITL^c formula over an extended set of propositions, whose set of models is the same as the set of models of the original formula, modulo a projection to the original set of propositions followed by a truncation of a fixed length prefix from the models. In particular the transformation is satisfiability-preserving. As a result, we can conclude that the logic $\mathrm{MITL}^c_{S_I}$ remains decidable. This decidability result is not new as it follows from the work of Henzinger et. al. in [8] where they show that Recursive Event-clock logic is equal in expressiveness to $\mathrm{MITL}^c_{S_I}$ in the continuous semantics. Nonetheless, our construction gives a different and a more direct proof of this fact.

We should point out here that our translations only preserve the equivalence of models up to projection onto the original set of propositions (and hence satisfiability), and not expressiveness in general. In fact, each of the logics MTL^{pw}, MTL^c, and MITL^c are known to be be strictly less expressive than their counterparts with the S operator [2, 14]. However, this fact together with our elimination results, tells us something about the class of languages definable in these logics: namely, that none of the logics MTL^{pw}, MTL^c, and MITL^c are closed under the operation of projection.

It also follows that the class of languages definable by $\mathrm{MITL}^c_{S_I}$ are contained in the class of languages definable by (continuous time) Alur-Dill timed automata. This follows since MITL^c was shown to be translatable to the class of continuous timed automata [1], which in turn are closed under projection. In a similar way, it also follows that MTL^{pw}_S over finite words is contained the class of languages definable by 1-clock alternating timed automata.

In related research, the well-known work of [7] shows how to eliminate S from pointwise classical LTL, without expanding the set of propositions, thus preserving expressiveness in addition to satisfiability. However to the best of our knowledge, no similar result is known for continuous time. The elimination of S can also be seen to follow from the connection between finite-state automata and monadic second order (MSO) logics, in both pointwise and continuous time ([3, 4]). This is because one can go from LTL with S to MSO (in fact its first-order fragment), then to automata, and then back to an existentially quantified LTL formula (without S).

In another related piece of work Hirshfeld and Rabinovich [9] show how to eliminate the future distance operator $\Diamond_I$ (with I non-singular) using existentially quantified "timer" formulas which can express the past distance operator $\diamondminus_I$.

In the rest of this paper we concentrate on the continuous semantics. The pointwise case is similar and easier to handle. Further details can be found in the technical report [5].

2 MTL in the continuous semantics

We denote the set of non-negative real numbers by $\mathbb{R}_{\geq 0}$. We use the standard notation to represent intervals, which are convex subsets of $\mathbb{R}_{\geq 0}$. For example $[2, \infty)$ denotes the set $\{t \in \mathbb{R}_{\geq 0} \mid 2 \leq t\}$. We use $\mathcal{I}_{\mathbb{Q}}$ to denote the set of intervals whose bounds are either rational or ∞. For any interval $I \in \mathcal{I}_{\mathbb{Q}}$, let $l(I)$ be the left limit and $r(I)$ be the right limit of I respectively. Then we denote the *length of* I, i.e. $r(I) - l(I)$ by $len(I)$. We also denote by $t + I$ the interval I' such that $t' \in I'$ iff $t' - t \in I$.

In the continuous semantics MTL is typically interpreted over "timed state sequences". Before we define these, let us first introduce the notion of a finitely varying function. Let B be a finite non-empty set, and let $f : \mathbb{R}_{\geq 0} \to B$. Then $t \in \mathbb{R}_{\geq 0}$ is a point of *left discontinuity* for f if there does not exist an $\epsilon > 0$ such that f is constant in the interval $(t - \epsilon, t]$. Similarly, $t \in \mathbb{R}_{\geq 0}$ is a point of *right discontinuity* for f if there does not exist an $\epsilon > 0$ such that f is constant in the interval $[t, t + \epsilon)$. The point t is a point of *discontinuity* for f if it is a point of left or right discontinuity. The function f is called *finitely varying* if it has only a finite number of points of discontinuity in any bounded interval in its domain.

Let P be a finite set of propositions. A *timed state sequence* τ over P is a finitely varying map $\tau : \mathbb{R}_{\geq 0} \to 2^P$. An equivalent definition (as given in [1]) is that there exists a sequence of subsets of propositions $s_0, s_1, \ldots$ and a sequence of intervals $I_0, I_1, \ldots$ satisfying:

1. I_0 is of the form $[0, r\rangle$ for some $r \in \mathbb{R}_{\geq 0}$ where we use '$\rangle$' to stand for the bracket '$)$' or '$]$'.
2. Every pair of intervals I_j and I_{j+1} are *adjacent* in the sense that I_j and I_{j+1} are disjoint, and $I_j \cup I_{j+1}$ forms an interval.
3. The sequence of intervals is "progressive" in that for every $t \in \mathbb{R}_{\geq 0}$, there exists $j \in \mathbb{N}$ such that $t \in I_j$.
4. The function τ is constant and equal to s_j in each I_j.

We call the sequence $(s_0, I_0)(s_1, I_1) \cdots$ above an *interval representation* of the function τ. It is easy to see that a timed state sequence τ

has a "canonical" interval representation of the form $(s_0, I_0)(s_1, I_1)\cdots$ where the I_i's are an alternating sequence of singular and open intervals (i.e. for each i, I_{2i} is of the form $[l, l]$ and I_{2i+1} is of the form (l, r)), where the singular intervals are precisely the points of discontinuity of τ. We denote the set of timed state sequences over P by $TSS(P)$.

The continuous version of MTL will be denoted by MTL^c. The syntax of MTL^c formulas over a set of propositions P is given by:

$$\varphi ::= \; p \mid \varphi U_I \varphi \mid \neg\varphi \mid \varphi \vee \varphi,$$

where $p \in P$ and I is an interval in $\mathcal{I}_{\mathbb{Q}}$.

The formulas of MTL^c above are interpreted over timed state sequences over P. Let τ be a timed state sequence over P, and let $t \in \mathbb{R}_{\geq 0}$. Then the satisfaction relation $\tau, t \models \varphi$ is given by:

$$\begin{array}{lll} \tau, t \models p & \text{iff} & p \in \tau(t) \\ \tau, t \models \psi U_I \eta & \text{iff} & \exists t' \geq t : \tau, t' \models \eta, \; t' - t \in I, \text{ and} \\ & & \forall t'' : t < t'' < t', \; \tau, t'' \models \psi \\ \tau, t \models \neg\psi & \text{iff} & \tau, t \not\models \psi \\ \tau, t \models \psi \vee \eta & \text{iff} & \tau, t \models \psi \text{ or } \tau, t \models \eta. \end{array}$$

We say that a timed word τ satisfies a MTL^c formula φ, written $\tau \models \varphi$, if and only if $\tau, 0 \models \varphi$, and set $L(\varphi) = \{\tau \in TSS(P) \mid \tau \models \varphi\}$.

We can also consider a version of MTL with the past modality S_I whose semantics is given by:

$$\begin{array}{lll} \tau, t \models \psi S_I \eta & \text{iff} & \exists t' \leq t : \tau, t' \models \eta, \; t - t' \in I, \text{ and} \\ & & \forall t'' : t' < t'' < t, \; \tau, t'' \models \psi. \end{array}$$

We denote this logic by MTL$^c_{S_I}$.

We define the standard temporal abbreviations as follows: $\psi U \eta \equiv \psi U_{[0,\infty)} \eta$, $\psi S \eta \equiv \psi S_{[0,\infty)} \eta$, $\Diamond\psi \equiv \top U \psi$, $\Box\psi \equiv \neg\Diamond\neg\psi$, $\Diamond_I \psi \equiv \top U_I \psi$, $\Box_I \psi \equiv \neg\Diamond_I \neg\psi$.

It will be convenient to work with a slightly different presentation of MTL. Let us define a base logic which is similar to classical continuous time LTL [10], and which we denote by LTL^c. The syntax of the logic (over the set of propositions P) is given by $\varphi ::= p \mid \varphi U^s \varphi \mid \neg\varphi \mid \varphi \vee \varphi$, and is interpreted over timed words in the same way as MTL^c above, with the modality U^s interpreted as $U_{(0,\infty)}$. Thus U^s is a "strict" until modality, which is strict in both its arguments. This is the natural choice for the until modality in continuous time and in the absence of an interval constraint. We note that the non-strict modality U is expressible using U^s, as $\psi U \eta \equiv \eta \vee (\psi U^s \eta)$, but not vice-versa. We define the derived modalities $\Diamond^s\psi$ and $\Box^s\psi$ to be: $\Diamond^s\psi \equiv \top U^s \psi$ and $\Box^s\psi \equiv \neg(\Diamond^s\neg\psi)$.

To this base logic we can add the past-time modalities S^s (for "strict since") and the distance operators $\diamondsuit$ and $\diamondminus$ to get the logic $\mathrm{LTL}^c(S^s, \diamondsuit, \diamondminus)$, whose syntax is given by

$$\varphi ::= \; p \mid \varphi U^s \varphi \mid \neg\varphi \mid \varphi \vee \varphi \mid \varphi S^s \varphi \mid \diamondsuit_I \varphi \mid \diamondminus_I \varphi.$$

The modality S^s is interpreted as $S_{(0,\infty)}$. We also denote the "non-recursive" versions of these logics by nr-LTL^c (with the appropriate arguments), in which we restrict the use of the distance operators to the propositions in P, i.e. we allow only distance subformulas of the form $\diamondsuit_I p$ and $\diamondminus_I p$, with $p \in P$.

The logic MTL^c can be seen to be expressively equivalent to the logic $\mathrm{LTL}^c(\diamondsuit)$ as the U_I modality of MTL^c can be expressed in terms of U^s and $\diamondsuit$. For example, if $I = [l, l]$, then $\psi U_I \eta = (\Box_{(0,l)}\psi) \wedge (\diamondsuit_{[l,l]}\eta)$; and if $I = (l, r)$ then $\psi U_I \eta = (\Box_{(0,l]}\psi) \wedge (\diamondsuit_{[l,l]}(\psi U \eta)) \wedge (\diamondsuit_{(l,r)}\eta)$. Similarly, the logic $\mathrm{MTL}^c_{S_I}$ can be seen to be equivalent to the logic $\mathrm{LTL}^c(S^s, \diamondsuit, \diamondminus)$.

Below we give some definitions which we will use in later sections. For any formula φ in $\mathrm{LTL}^c(S^s, \diamondsuit, \diamondminus)$ and a timed state sequence τ we define the characteristic function for φ in τ, $f_{\varphi,\tau} : \mathbb{R}_{\geq 0} \to \{\top, \bot\}$, given by $f_{\varphi,\tau}(t) = \top$ if $\tau, t \models \varphi$ and $\bot$ otherwise. We note that the function $f_{\varphi,\tau}$ is a finitely varying function. This follows from the argument in [1] which says that for every timed state sequence τ and every $\mathrm{LTL}^c(\diamondsuit)$ formula φ, there is an equivalent interval representation of τ (i.e. denoting the same function as τ) which is "φ-fine" – i.e. φ is constant throughout each interval in the interval sequence. We say that a point t is a point of (right) discontinuity in τ w.r.t. the formula φ, if it is a point of (right) discontinuity of the function $f_{\varphi,\tau}$.

As an example of what we can say in the base logic LTL^c, we define the "macro" formula $rd(\varphi)$ that will be of use later in the paper, which characterises points in a timed state sequence at which φ is true and which are points of right discontinuities w.r.t. φ. We define $rd(\varphi) = \varphi \wedge ((\neg\varphi) U^s (\neg\varphi))$.

3 Flattening MTL^c

We now show that each of the sublogics of $\mathrm{LTL}^c(S^s, \diamondsuit, \diamondminus)$ can be flattened to its non-recursive version. We show that every $\mathrm{LTL}^c(\diamondsuit)$ formula (equivalently an MTL^c formula) over a set of propositions P, can be flattened to an nr-$\mathrm{LTL}^c(\diamondsuit)$ formula over a set of propositions P' which is an extension of P. We assume the standard notion of subformulas: thus the subformulas of the formula $\varphi = pU(\diamondsuit_{[1,2]}(q \vee \diamondsuit_{(0,\infty)} r))$ are φ, p, $\diamondsuit_{[1,2]}(q \vee \diamondsuit_{(0,\infty)} r)$, $q \vee \diamondsuit_{(0,\infty)} r$, q, $\diamondsuit_{(0,\infty)} r$ and r. The *distance subformulas* of a formula are all its subformulas of the form $\diamondsuit_I \psi$.

We define the *level* of an $\mathrm{LTL}^c(\Diamond)$ formula φ as a measure of the nesting depth of distance subformulas in φ. Inductively, the level of a formula without any distance subformulas is 0; the level of a formula φ is $i+1$ if it has a distance subformula of the form $\Diamond_I\psi$ with ψ a level i formula, and no distance subformula of the form $\Diamond_J\eta$ with the level of η more than i. A *top-level* distance subformula of φ is a distance subformula which has at least one occurrence outside the scope of any other distance subformula. More formally, the set of top-level distance subformulas of φ, denoted *top-dsf*(φ) is defined inductively as:

$$\begin{array}{lcl} \textit{top-dsf}(p) & = & \{\} \\ \textit{top-dsf}(\psi U^s \eta) & = & \textit{top-dsf}(\psi) \cup \textit{top-dsf}(\eta) \\ \textit{top-dsf}(\psi \vee \eta) & = & \textit{top-dsf}(\psi) \cup \textit{top-dsf}(\eta) \\ \textit{top-dsf}(\Diamond_I \psi) & = & \{\Diamond_I \psi\}. \end{array}$$

Let us call a set of formulas X *closed* if for every distance subformula $\Diamond_I\psi$ of a formula in X, the formula ψ also belongs to X.

We fix a set of propositions P for the rest of this section.

Let $X = \{\psi_0, \ldots, \psi_n\}$ be a closed set of formulas, in increasing order of level. Let $\tau = (s_0, I_0)(s_1, I_1)\cdots$ be a timed state sequence over P. Let P' be the extended set of propositions $P \cup \{p_0, \ldots, p_n\}$, where each proposition p_j is meant to capture the truth of ψ_j. Then we define the *canonical extension* of τ w.r.t. X, denoted by *can-ext*$_X(\tau)$, to be the timed state sequence over P', given by *can-ext*$_X(\tau)(t) = \tau(t) \cup \{p_j \mid \tau, t \models \psi_j\}$.

We need to argue that *can-ext*$_X(\tau)$ is indeed a timed state sequence, in that it is finitely varying. But this is true since as observed earlier, the characteristic function $f_{\tau,\psi}$, for each ψ in τ is finitely varying.

For a timed state sequence τ' over P', let us denote the timed state sequence over P obtained by projecting each $\tau'(t)$ to P by $\tau'\lceil P$. Thus $(\tau'\lceil P)(t) = \tau'(t) \cap P$. We extend the notion of projection to a set of timed state sequences over P' in the natural way.

As an example, consider the formula $\varphi = \Diamond_{(0,1)}\psi_2$ where $\psi_2 = \Diamond_{[0,1]}\psi_1$ and $\psi_1 = p \wedge \Diamond_{[1,1]}q$. Let us take X to be the set $\{q, \psi_1, \psi_2\}$. Then the canonical extension of the timed state sequence

$$\tau = (\{p\}, [0, 1.5))(\{q\}, [1.5, 2])(\{p, q\}, (2, \infty))$$

with respect to X is shown in Fig. 1.

Let us now define the "canonical translation" of a formula φ. Let X be any closed finite set of formulas containing the set of formulas ψ such that φ has a distance subformula of the form $\Diamond_I\psi$. Then the *canonical translation* of φ w.r.t. X, denoted *can-tr*$_X(\varphi)$, is obtained from φ by replacing every top-level distance subformula of the form

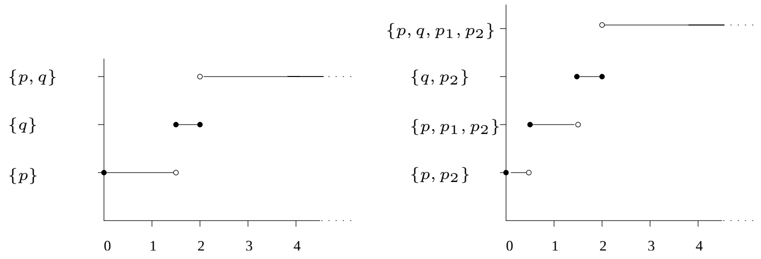


Figure 1: TSS τ and its canonical extension w.r.t. X.

$\Diamond_I \psi_k$ by $\Diamond_I p_k$. We note that for any φ, $\mathit{can}\text{-}\mathit{tr}_X(\varphi)$ is always a formula in nr-LTL$^c(\Diamond)$ over P'.

Lemma 1. *Let φ and X be as above, and let τ be a timed state sequence over P. Then $\tau, t \models \varphi$ iff $\mathit{can}\text{-}\mathit{ext}_X(\tau), t \models \mathit{can}\text{-}\mathit{tr}_X(\varphi)$.*

Proof. The proof follows easily by induction on the structure of φ. For the base case, clearly $\tau, t \models p$ iff $\mathit{can}\text{-}\mathit{ext}_X(\tau), t \models p$. For the inductive step, the only interesting case is $\Diamond_I \psi_j$. But $\tau, t \models \Diamond_I \psi_j$ iff $\mathit{can}\text{-}\mathit{ext}_X(\tau), t \models \Diamond_I p_j$, since ψ_j is true in τ precisely when the proposition p_j is true in $\mathit{can}\text{-}\mathit{ext}_X(\tau)$. ∎

We show that the property of an extended timed state sequence being a canonical extension of a timed state sequence w.r.t. a closed set of formulas X is definable in nr-LTL$^c(\Diamond)$. Let $X = \{\psi_0, \ldots, \psi_n\}$ be a closed set of formulas, with $\psi_0, \ldots, \psi_n$ being in increasing order of level. The formula ν^X_{can} basically says that the value of each p_i at a point in the extended word is correct. We define

$$\nu^X_{\mathit{can}} = \Box\left(\bigwedge_{j=0}^{n} (p_j \Leftrightarrow \mathit{can}\text{-}\mathit{tr}_X(\psi_j))\right).$$

Lemma 2. *Let X be as above. Let $P' = P \cup \{p_0, \ldots, p_n\}$ and let τ' be a timed state sequence over P'. Then τ' is the canonical extension of $\tau' \restriction P$ w.r.t. X iff $\tau' \models \nu^X_{\mathit{can}}$.*

Proof. Let τ' be a canonical extension of $\tau' \restriction P$. Then we can argue by induction on j going from 0 to n that $\tau' \restriction P, t \models \psi_j$ iff $\tau', t \models \mathit{can}\text{-}\mathit{tr}_X(\psi_j)$. This gives us that $\tau' \models \nu^X_{\mathit{can}}$.

Conversely, suppose $\tau' \models \nu^X_{\mathit{can}}$. Then once again we argue by induction on j going from 0 to n that $\tau' \restriction P, t \models \psi_j$ iff $\tau', t \models \mathit{can}\text{-}\mathit{tr}_X(\psi_j)$. From the definition of ν^X_{can} we then have that $\tau', t \models \mathit{can}\text{-}\mathit{tr}_X(\psi_j)$ iff $\tau', t \models p_j$. Hence $\tau' \restriction P, t \models \psi_j$ iff $\tau', t \models p_j$, and therefore τ' is a canonical extension w.r.t. X. ∎

We can now construct the required flattening of a formula φ in $\mathrm{LTL}^c(\Diamond)$. Let X be the set of all ψ such that there is a distance subformula of φ of the form $\Diamond_I \psi$. Note that X is closed. We now define φ' to be $\mathit{can}\text{-}\mathit{tr}_X(\varphi) \wedge \nu^X_{can}$. By Lemmas 1 and 2, it follows that $L(\varphi) = L(\varphi') \upharpoonright P$. To summarize:

Theorem 3. *Let $\varphi \in \mathrm{LTL}^c(\Diamond)$ over the set of propositions P. Then we can construct a formula $\varphi' \in nr\text{-}\mathrm{LTL}^c(\Diamond)$ over an extended set of propositions P', such that $L(\varphi) = L(\varphi') \upharpoonright P$. In particular φ is satisfiable iff φ' is.* ∎

We also observe that the flattening carried out here can be extended to the past operator $\diamondminus$, as well as the S modality, meaning that given a formula φ we give a projection equivalent formula φ' over P' in which only S formulas are of the form pSq, where $p, q \in P'$.

4 Eliminating S^s in LTL^c

In this section we show how to go from a formula in $\mathrm{LTL}^c(S^s)$ to a satisfiability-equivalent formula in LTL^c over an extended alphabet.

Let φ be an $\mathrm{LTL}^c(S^s)$ formula over the set of propositions P. Without loss of generality we assume that φ has been flattened, and thus the only occurrences of S^s formulas are of the form pS^sq with $p, q \in P$. The aim is to eliminate the formula pS^sq by introducing a new proposition r and adding formulas which make sure that r holds precisely at the points where pS^sq holds. The idea is to consider intervals in the model in which the truth of p and q is constant and to ensure that r is updated correctly in these intervals based on the values of p and q in the intervals.

Let $\tau \in TSS(P)$ and let t be a point in τ. Let $t' > t$ be the next point of discontinuity in τ w.r.t. p and q (i.e. a point of discontinuity of either $f_{p,\tau}$ or $f_{q,\tau}$), if such a point exists. Then Table 1 summarizes the truth value of the formula pS^sq in the interval (t, t') and at t', depending on the values of pS^sq, p and q at t, and the values of p and q in the interval (t, t'). The table for the case when there does not exist such a point t' is similar.

No.	t	p, q in (t, t')	pS^sq in (t, t')	pS^sq at t'
1	δ	$p \wedge q$	$true$	$true$
2	$(((pS^sq) \wedge p) \vee q)$	$p \wedge \neg q$	$true$	$true$
3	$\neg(((pS^sq) \wedge p) \vee q)$	$p \wedge \neg q$	$false$	$false$
4	δ	$\neg p \wedge \neg q$	$false$	$false$
5	δ	$\neg p \wedge q$	$false$	$false$

Table 1: The value of pS^sq in (t, t') and at t'.

The entries in Table 1 can be explained as follows. The δ in the table represents "don't care". If $p \wedge q$ is true throughout the interval (t, t') then clearly $pS^s q$ is true throughout the interval (t, t') and also at t'. If $p \wedge \neg q$ is true throughout the interval (t, t') then the truth value of $pS^s q$ in the interval depends on the truth values of $pS^s q$, p and q at t. It is easy to see that if q is true at t then $pS^s q$ is true everywhere in the interval (t, t') and at t'. Similarly if $(pS^s q) \wedge p$ is true at t then $pS^s q$ is true throughout the interval (t, t') and at t'. The remaining entries can be similarly explained.

Now let $X = \{p_0 S^s q_0, \ldots, p_n S^s q_n\}$ be all the S^s subformulas in φ. We introduce a proposition r_j for each $p_j S^s q_j$ which is meant to capture the truth of $p_j S^s q_j$, and call this extended set of propositions P'. The canonical extension of τ in $TSS(P)$ is τ' in $TSS(P')$, given by $\tau'(t) = \tau(t) \cup \{r_j \mid \tau, t \models p_j S^s q_j\}$. The translation of φ obtained by replacing every occurrence of $p_j S^s q_j$ by r_j, clearly preserves the truth of φ in the canonical extension. It is now sufficient if we can define a formula ν_{can} in LTL^c over P' which describes precisely the timed state sequences over P' which are canonical extensions w.r.t. X.

It will be convenient to use the macro $\alpha(\psi, \mu)$ defined below, which is true at a point t in a model τ iff either there is a subsequent point of discontinuity t' in τ w.r.t. ψ such that ψ is true throughout the interval (t, t'), and μ is true throughout the interval $(t, t']$; or ψ is true throughout (t, ∞) and so is μ. We define $\alpha(\psi, \mu)$ to be:

$$((\psi U^s((\neg\psi) \vee rd(\psi))) \Rightarrow \\ ((\psi \wedge \mu) U^s(((\neg\psi) \vee rd(\psi)) \wedge \mu))) \wedge ((\Box^s \psi) \Rightarrow (\Box^s \mu)).$$

We now define ν_{can} as follows:

$$\nu_{can} = \bigwedge_{j=0}^{n} ((\neg r_j) \wedge \Box(\varphi_{j1} \wedge \varphi_{j2} \wedge \varphi_{j3} \wedge \varphi_{j4} \wedge \varphi_{j5}))$$

where

1. $\varphi_{j1} : \alpha(p_j \wedge q_j, r_j)$
2. $\varphi_{j2} : ((r_j \wedge p_j) \vee q_j) \Rightarrow \alpha(p_j \wedge \neg q_j, r_j)$
3. $\varphi_{j3} : \neg((r_j \wedge p_j) \vee q_j) \Rightarrow \alpha(p_j \wedge \neg q_j, \neg r_j)$
4. $\varphi_{j4} : \alpha(\neg p_j \wedge \neg q_j, \neg r_j)$
5. $\varphi_{j5} : \alpha(\neg p_j \wedge q_j, \neg r_j)$

Lemma 4. *Let $\tau' \in TSS(P')$. Then $\tau' \models \nu_{can}$ iff τ' is a canonical extension w.r.t. X.*

Proof. Let τ' be a canonical extension w.r.t. X. Then we argue that τ' satisfies ν_{can}. Consider any $p_j S^s q_j$ in X. Since $p_j S^s q_j$ is not satisfied at time 0 in τ', we have $\neg r_j$ true at time 0 in τ'. To show that the other conjuncts in ν_{can} are satisfied, let t be a point in τ'. Now, there are two cases: either there is a next point of discontinuity t' of p_j and q_j in τ' after t, or there is none. Let us consider the first case. Then exactly one of $p_j \wedge q_j$, $p_j \wedge \neg q_j$, $\neg p_j \wedge \neg q_j$, $\neg p_j \wedge q_j$ is true throughout the interval (t, t'). Say for example that $p_j \wedge q_j$ was true. Then, from the table 1, we must have $\alpha(p_j \wedge q_j, r_j)$ true at t, and hence φ_{j1} is satisfied at time t in τ'. The formulas φ_{j2} to φ_{j5} are vacuously satisfied at t. Similarly, the other cases can be handled. The second case when there is no point of discontinuity after t is also handled similarly.

For the converse direction let $\tau' \models \nu_{can}$. We argue that τ' is a canonical extension w.r.t. X, i.e for every $p_j S^s q_j \in X$ and for every $t \in \mathbb{R}_{\geq 0}, \tau', t \models r_j$ iff $\tau', t \models p_j S^s q_j$. Let $t_0, t_1, \ldots$ be the (finite or infinite) sequence of discontinuities in τ' w.r.t. p_j and q_j. Let $I_0 = [t_0, t_0]$, $I_1 = (t_0, t_1]$, $I_2 = (t_1, t_2]$ and so on. We use induction on i to prove that the value of r_j is correctly updated in the interval I_i.

Base case: Since $\tau' \models \nu_{can}$, at time 0 we have r_j is not true. Since $p_j S^s r_j$ is also not true at time 0, the value of r_j is correctly updated in I_0.

Induction step: Let us assume that r_j is correctly updated in all the intervals up to I_i. We now argue that for all $t \in I_{i+1}$ we have $\tau', t \models p_j S^s q_j$ iff $\tau', t \models r_j$. There are two cases: either there exists a point of discontinuity t_{i+1}, or $I_{i+1} = (t_i, \infty)$.

For the case there exists t_{i+1}: Then exactly one these $p_j \wedge q_j$, $p_j \wedge \neg q_j$, $\neg p_j \wedge \neg q_j$, $\neg p_j \wedge q_j$ holds in (t_i, t_{i+1}). According to the table 1 the value of $p_j S^s q_j$ is fixed in the interval I_{i+1}. Further, by the induction hypothesis the value of r_j is correctly updated at the point t_i. Hence, the formulas φ_{j1} to φ_{j5} ensure that the value of r_j is correctly updated in I_{i+1}.

For the case there does not exist t_{i+1}, the argument is again similar. ∎

5 Eliminating $\diamondminus$ in LTL^c

We now show how we can remove the past distance modality $\diamondminus$ from the continuous version of MTL, while preserving satisfiability of the original formula.

Let φ be a formula in any of the versions of our continuous logic LTL^c with the $\diamondminus$ operator. By the results of Section 3 we can assume the $\diamondminus$ subformulas in φ are of the form $\diamondminus_I p$ with p a proposition in P. We essentially show how to express formulas of the form $\diamondminus_I p$ in terms of new propositions, the U modality, and $\diamond$ operator.

Let $\diamondminus_{I_0} p_0, \ldots, \diamondminus_{I_n} p_n$ be all the $\diamondminus_I$ subformulas in φ. Let τ be a timed state sequence over P. Let P' be the extended set of propositions $P \cup \{q_0, \ldots, q_n\} \cup \{r_0, \ldots, r_n\}$, where q_j and r_j are new propositions associated with each $\diamondminus_{I_j} p_j$. The proposition q_j is meant to capture the truth of $\diamondminus_{I_j} p_j$, and r_j is meant to capture the fact that we have seen a p_j sometime strictly in the past. We define the canonical extension of τ (w.r.t. $\{\diamondminus_{I_0} p_0, \ldots, \diamondminus_{I_n} p_n\}$) to be the timed state sequence τ' over P', given by

$$\tau'(t) = \tau(t) \cup \{q_j \mid \tau, t \models \diamondminus_{I_j} p_j\} \cup \{r_j \mid \exists t' < t : \; \tau, t' \models p_j\}.$$

The canonical translation of φ is obtained by simply replacing each $\diamondminus_{I_j} p_j$ by q_j. It is clear that $\tau, t \models \varphi$ iff the canonical translation of φ is satisfied at t in the canonical extension of τ.

We now define the formula ν_{can} which characterises canonical extensions. The formula ν_{can} is the formula below

$$\bigwedge_{j=0}^{n} ((\neg r_j) \wedge \Box(p_j \;\Rightarrow\; \Box_{(0,\infty)} r_j) \wedge \neg((\neg p_j) \wedge ((\neg p_j) U r_j)))$$

in conjunction with a formula ψ_j for each j as below:

- If I_j is of the form $[l, l]$ take ψ_j to be $\Box(p_j \Leftrightarrow \Diamond_{[l,l]} q_j)$;
- If I_j is of the form (l, r) take ψ_j to be $\Box((\Diamond_{[r,r]} q_j) \Leftrightarrow \Diamond_{(0,r-l)} p_j)$;
- If I_j is of the form (l, ∞) take ψ_j to be $\Box(r_j \Leftrightarrow \Diamond_{[l,l]} q_j)$;
- and if I_j is of the form $[l, \infty)$ take ψ_j to be $\Box((r_j \vee p_j) \Leftrightarrow \Diamond_{[l,l]} q_j)$.

We note that we have introduced the proposition r_j to avoid using a formula of the form $\top S p_j$ in the translation. The construction here is simpler than if we had used S and then eliminated it using the results of Section 4.

6 Eliminating $\diamondminus$ from $\mathrm{MITL}^c_{S_I}$

In this section we show that an $\mathrm{MITL}^c_{S_I}$ formula can be reduced to satisfiability equivalent $\mathrm{LTL}^c(\Diamond)$ formula. In section 2 it has been shown that an $\mathrm{MITL}^c_{S_I}$ formula can be reduced to an $\mathrm{LTL}^c(S^s, \Diamond, \diamondminus)$ formula. In section 3 we show that for given a recursive $\mathrm{LTL}^c(S^s, \Diamond, \diamondminus)$ formula we can construct a satisfiability equivalent non-recursive $\mathrm{LTL}^c(S^s, \Diamond, \diamondminus)$ formula over an extended alphabet. In section 4 we show that for given a non-recursive $\mathrm{LTL}^c(S^s, \Diamond, \diamondminus)$ formula we can construct a satisfiability equivalent non-recursive $\mathrm{LTL}^c(\Diamond, \diamondminus)$ formula, once again over an extended alphabet. We also note that the $\mathrm{LTL}^c(\Diamond, \diamondminus)$ formula so obtained also does not have any singular intervals since none of the above translations introduce singular intervals if the original formula is singular interval free.

We now show how to go from a formula φ in non-recursive $\mathrm{LTL}^c(\Diamond, \diamondminus)$ with non singular intervals to a satisfiability equivalent

LTL$^c(\diamond)$ formula over an extended set of propositions. Let $X = \{\diamondminus_{I_0} p_0, \ldots, \diamondminus_{I_n} p_n\}$ be the set of all past distance subformulas used by φ. The idea is introduce a proposition q_j for each $\diamondminus_{I_j} p_j \in X$ which is meant to be true precisely at the points were $\diamondminus_{I_j} p_j$ is true. Once again we define the canonical extension τ' of timed sequence $\tau \in TSS(P)$ w.r.t. X. We give a formula ν which characterises the canonical extensions of τ modulo a prefix. We also note that the formula ν does not introduce any singular intervals if the original formula is over non singular intervals.

We define ν as follows:

$$\nu = \bigwedge_{j=0}^{j=n} \nu_j$$

where ν_j makes sure that q_j is true precisely at the points were $\diamondminus_{I_j} p_j$ is true. The formula ν_j depends on whether I_j is left closed, right closed or unbounded. Below we give the formula ν_j for each case.

Case 1: If I_j is of the form (l, r) then we define ν_j as follows:

$$\nu_j = \Box(\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4 \wedge \varphi_5)$$

Then the formulas $\varphi_1, \ldots, \varphi_5$ are as given below:

1. $\varphi_1 : p_j \Rightarrow \Box_I q_j$
2. $\varphi_2 : \Box\neg q_j \vee (\neg q_j \Rightarrow \neg q_j U \Box_{(0,r-l)} q_j)$
3. $\varphi_3 : q_j \;\Rightarrow\; (q_j U \neg q_j) \vee \Box q_j$
4. $\varphi_4 : (\Box_{(l,r)} q_j \wedge \diamond_{[l,r)} \neg q_j) \Rightarrow p_j U p_j$
5. $\varphi_5 : \Box_{[r,2r-l)} q_j \Rightarrow (\diamond_{(0,r-l)} p_j \wedge \neg\diamond_{(0,r-l)} \Box_{(0,r-l)} \neg p_j)$.

Case 2: If I_j is of the form $[l, r)$ then we define ν_j as follows:

$$\nu_j = \Box(\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4 \wedge \varphi_5)$$

where the formulas $\varphi_1, \ldots, \varphi_5$ are as given below:

1. $\varphi_1 : p_j \Rightarrow \Box_I q_j$
2. $\varphi_2 : \Box\neg q_j \vee (\neg q_j \Rightarrow \neg q_j U \Box_{(0,r-l)} q_j)$
3. $\varphi_3 : q_j \;\Rightarrow\; (q_j U \neg q_j) \vee \Box q_j$
4. $\varphi_4 : (\Box_{(l,r)} q_j \wedge \diamond_{[l,r)} \neg q_j) \Rightarrow p_j U p_j$
5. $\varphi_5 : \Box_{[r,2r-l)} q_j \Rightarrow (\diamond_{(0,r-l]} p_j \wedge \neg\diamond_{(0,r-l)} \Box_{(0,r-l]} \neg p_j)$.

Case 3: If I_j is of the form $(l, r]$ then we define ν_j as follows:

$$\nu_j = \Box(\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4)$$

where the formulas $\varphi_1, \ldots, \varphi_4$ are as given below:

1. $\varphi_1 : p_j \Rightarrow \Box_I q_j$
2. $\varphi_2 : \Box\neg q_j \vee (\neg q_j \Rightarrow \neg q_j U \Box_{(0,r-l]} q_j)$
3. $\varphi_3 : (\Box_{(l,r]} q_j \wedge \Diamond_{[l,r]} \neg q_j) \Rightarrow p_j U p_j$
4. $\varphi_4 : \Box_{[r,2r-l]} q_j \Rightarrow (\Diamond_{[0,r-l)} p_j \wedge \neg\Diamond_{(0,r-l]} \Box_{[0,r-l)} \neg p_j)$.

Case 4: If I_j is of the form $[l,r]$ then we define ν_j as follows:

$$\nu = \Box(\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4)$$

where the formulas $\varphi_1, \ldots, \varphi_4$ are as given below:

1. $\varphi_1 : p_j \Rightarrow \Box_I q_j$
2. $\varphi_2 : \Box\neg q_j \vee (\neg q_j \Rightarrow \neg q_j U \Box_{(0,r-l]} q_j)$
3. $\varphi_3 : (\Box_{(l,r]} q_j \wedge \Diamond_{[l,r]} \neg q_j) \Rightarrow p_j U p_j$
4. $\varphi_4 : \Box_{[r,2r-l]} q_j \Rightarrow (\Diamond_{[0,r-l]} p_j \wedge \neg\Diamond_{(0,r-l]} \Box_{[0,r-l]} \neg p_j)$.

In the next two cases we handle the unbounded cases.
Case 5: If I_j is of the form (l,∞) then we define ν_j as follows:

$$\nu_j = \Box(\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4)$$

where the formulas $\varphi_1, \ldots, \varphi_4$ are as given below:

1. $\varphi_1 : p_j \Rightarrow \Box_I q_j$
2. $\varphi_2 : \Box\neg q_j \vee (\neg q_j \Rightarrow \neg q_j U \Box_{(0,\infty)} q_j)$
3. $\varphi_3 : (\Box_{(l,\infty)} q_j \wedge \Diamond_{[l,\infty)} \neg q_j) \Rightarrow p_j U p_j$
4. $\varphi_4 : \neg(\neg q_j U \Box_{[l,\infty)} q_j)$

Case 6: If I_j is of the form $[l,\infty)$ then we define ν_j as follows:

$$\nu_j = \Box(\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4)$$

where the formulas $\varphi_1, \ldots, \varphi_4$ are as given below:

1. $\varphi_1 : p_j \Rightarrow \Box_I q_j$
2. $\varphi_2 : \Box\neg q_j \vee (\neg q_j \Rightarrow \neg q_j U \Box_{(0,\infty)} q_j)$
3. $\varphi_3 : (\Box_{(l,\infty)} q_j \wedge \Diamond_{[l,\infty)} \neg q_j) \Rightarrow p_j U p_j$
4. $\varphi_4 : (\neg q_j U \Box_{[l,\infty)} q_j) \Rightarrow p_j$

For each $1 \leq j \leq n$, we define d_j as follows:

$$d_j = \begin{cases} 2r(I_j) - l(I_j) & \text{if } r(I_j) < \infty \\ l(I_j) & \text{otherwise.} \end{cases}$$

Let d_{max} be that the maximum value among all the d_j and let $P' = P \cup \{q_j | 1 \leq j \leq n\}$.

Lemma 5. *Let $\tau' \in TSS(P')$. If τ' is a canonical extension of τ w.r.t. $\diamondminus_{I_j} p_j$, then $\tau' \models \nu_j$. Conversely if $\tau' \models \nu_j$, then for all $t \geq d_{max}$, $\tau', t \models \diamondminus_{I_j} p_j$ iff $\tau', t \models q_j$.*

Proof. For convenience we assume that I_j is of the form (l, r). We can prove the lemma for other cases along the similar lines.

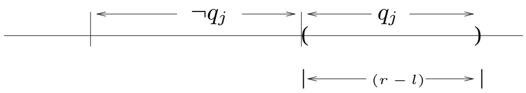
Let τ' be a canonical extension w.r.t. $\diamondminus_{I_j} p_j$ and let t be a point in τ'. Then we argue that τ at t satisfies the formulas $\varphi_1, \ldots, \varphi_5$. Hence it follows that $\tau' \models \nu_j$. We now explain the formulas $\varphi_1, \ldots, \varphi_5$ in the proof given below. These formulas say that ν_j characterises the canonical extensions w.r.t. $\diamondminus_{I_j} p_j$ except possibly for a prefix.

Let p_j be true at t in τ'. Then $\diamondminus_{I_j} p_j$ is true in the interval $t + I_j$. Since τ' is a canonical extension w.r.t. $\diamondminus_{I_j} p_j$ we have that q_j is true throughout the interval $t + I_j$. Hence $\tau', t \models \varphi_1$. Pictorially:

Let t be a point such that $\tau', t \models \neg q_j$. Then we have two cases:

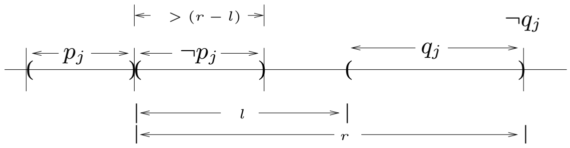
Case 1: $\neg q_j$ true always in the interval $[t, \infty)$ in which case $\tau', t \models \Box \neg q_j$.

Case 2: There exists a $t' > t$ such that $\tau', t' \models q_j$. Since τ' is a canonical extension it should be the case that there exists a point t'' in τ' such that $\tau', t'' \models p_j$ and $t' \in t'' + I_j$. Since $\tau', t \models \neg q_j$ and $\tau', t \models \varphi_1$, we can easily argue that $t < t'' + l(I_j)$. Since $\diamondminus_{I_j} p_j$ is true throughout the interval $t'' + I_j$ we have $\tau', t \models \varphi_2$.



For any formula ψ in LTL$^c(\Diamond, \diamondminus)$ and a timed state sequence τ, we say that an interval J is *ψ-interval* in τ iff for each $t \in J$, $f_{\tau,\psi}(t) = \top$ and the interval J is said to be a *maximal ψ-interval* in τ iff there does not exist an interval J' such that $J \subset J'$ and for each $t' \in J'$, $f_{\tau,\psi}(t') = \top$.

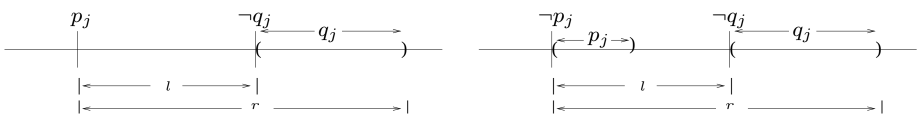
Let $\tau', t \models q_j$. We now argue that $\tau', t \models \varphi_3$. So let J be the maximal q_j-interval such that $t \in J$. Since the interval I_j is both left and right open, any maximal $\diamondminus_{I_j} p_j$-interval should be both left and right open. Since τ' is a canonical extension w.r.t. $\diamondminus_{I_j} p_j$ we have that any maximal q_j-interval is also both left and right open and therefore $\tau', t \models \varphi_3$. Pictorially:



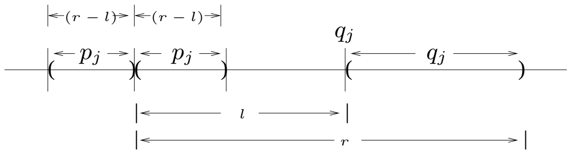
We argue that $\tau', t \models \varphi_4 \wedge \varphi_5$. Let $\tau', t \models q_j$. Since $\tau', t \models \varphi_1 \wedge \varphi_2 \wedge \varphi_3$ we can easily argue that there exists an q_j-interval J, which is both left and right open, such that $t \in J$ and $len(J) = len(I_j) = r - l$.

Now there are two cases:

Case 1: Let $\tau', l(J) \models \neg q_j$ and let $t' = l(J) - l$. Then the formula φ_4 is applicable only if $t' \geq 0$, so we assume this is so. Then we have that $\tau', t' \models \Box_{(l,r)} q_j \wedge \Diamond_{[l,r)} q_j$. We now argue that $\tau', t' \models p_j U p_j$. Suppose $\tau', t' \not\models p_j U p_j$. This implies that there exists an $\epsilon > 0$ such that for every $t'' \in [t', t' + \epsilon)$, $\tau', t'' \models \neg p_j$. Since $\neg q_j$ is true at $l(J)$ it follows that p_j is not true in the $(t' - (r - l), t')$ and therefore everywhere in the interval $(t' - (r - l), t' + \epsilon)$ we have $\neg p_j$ true. But this implies q_j is not true throughout in the interval J, a contradiction. The two cases where the formula $p_j U p_j$ is true at t' are shown side by side in the figure given below:



Case 2: Let $\tau', l(J) \models q_j$ and let $t' = l(J) - r$. Then the formula φ_5 is applicable only if $t' \geq 0$, so we assume this is so. Then we have that $\tau', t' \models \Box_{[r, 2r-l)} q_j$. We now argue $\tau', t' \models \Diamond_{(0,r-l)} p_j \wedge \neg \Diamond_{(0,r-l)} \Box_{(0,r-l)} \neg p_j$. Since q_j is true at $l(J)$ it should be that case that there exists a p_j in the interval $(l(J) - r, l(J) - l)$, i.e. in the interval $(t', t' + (r - l))$. Hence $\tau', t' \models \Diamond_{(0,r-l)} p_j$. Again if we have a point $t'' \in (t', t' + (r - l))$ such that $\tau', t'' \models \Box_{(0,r-l)} \neg p_j$ then we have that $\neg q_j$ is true at the point $t'' + r$. Since $l(J) - r < t'' < l(J) - l$, we have $l(J) < t'' + r < r(J)$, a contradiction. Pictorially:



If $\tau', t \models \neg q_j$, then we can argue similarly that $\tau', t \models \varphi_4 \wedge \varphi_5$.

For the converse direction let $\tau' \models \nu$. We argue that for all $t \geq 2r - l$, $\tau', t \models \ominus_{I_j} p$ iff $\tau', t \models q$. By the first formula if $\tau', t \models \ominus_{I_j} p_j$ then $\tau', t \models q_j$. We now argue that if $\tau', t \models q_j$, then $\tau', t \models \ominus_{I_j} p_j$.

So let $\tau', t \models q_j$. Since $\tau', t \models \varphi_1 \wedge \varphi_2 \wedge \varphi_3$ we can easily argue that there exists a q_j-interval J, which is both left and right open, such that $t \in J$ and $len(J) = len(I_j) = r - l$. There are two cases:

1. $\tau', l(J) \not\models q_j$: This implies that $\tau', l(J) - l \models \Box_{(l,r)} q_j \wedge \Diamond_{[l,r)} \neg q_j$ (note that $t \geq 2r - l$, so $l(J) - l > 0$). Since $\tau' \models \varphi_4$ it follows that $\tau', l(J) - l \models p_j U p_j$ and therefore $\tau', t \models \diamondminus_{I_j} p_j$.
2. $\tau', l(J) \models q_j$: The proof is similar to the one given above which uses formula φ_5 to ensure that $\tau', t \models \diamondminus_{I_j} p_j$. $\blacksquare$

Corollary 6. *If $\tau \in TSS(P)$ is a model such that p_j is not true in the interval $[0, d)$, for any $d \geq 2r - l$, i.e. $p_j \notin \tau(t)$ for all $t \in [0, d)$ then $\tau' \in TSS(P')$ is a canonical extension of τ w.r.t. $\diamondminus_{I_j} p_j$ iff τ' satisfies $\nu \wedge \Box_{[0,d)} \neg q_j$.* $\blacksquare$

In order to take the advantage of Corollary 6 we first shift the models to the right. Let $\tau \in TSS(P)$ and let $d \geq 0$. Let c be a proposition such that $c \notin P$ and let $P' = P \cup \{c\}$. Then τ shifted by d time units to the right, called $\tau_d \in TSS(P')$, is defined as follows:

$$\tau_d(t) = \begin{cases} \tau(t-d) \cup \{c\} & \text{if } t \geq d \\ \phi & \text{otherwise.} \end{cases}$$

We define the *translated* version of a formula μ, called $tr(\mu)$, which essentially forces μ to be true at that points where c is true. $tr(\mu)$ is defined inductively as follows:

$$\begin{array}{lcl} tr(p) & = & p, \text{where } p \in P \\ tr(\neg\psi) & = & \neg tr(\psi) \\ tr(\psi \vee \eta) & = & tr(\psi) \vee tr(\eta) \\ tr(\psi U \eta) & = & tr(\psi) U tr(\eta) \\ tr(\Diamond_I \psi) & = & \Diamond_I (tr(\psi)) \\ tr(\diamondminus_I \psi) & = & \diamondminus_I (c \wedge tr(\psi)) \end{array}$$

Now we define the formula μ_d which is satisfied precisely by the models of μ shifted by d time units to the right:

$$\mu_d = (\Box_{[0,d)} \bigwedge_{p \in P} \neg p) \wedge (\Box_{[0,d)} \neg c) \wedge (\Box_{[d,\infty)} c) \wedge (\neg c U (c \wedge tr(\mu))).$$

For any timed state sequence τ and $d \geq 0$, let $tail_d(\tau)$ be the timed state sequence τ' such that $\forall t \in \mathbb{R}_{\geq 0}, \tau'(t) = \tau(d + t)$. We extend the function $tail_d$ in the natural way over the sets of timed state sequences.

Lemma 7. *Let μ be a formula. Then $L(\mu) = tail_d(L(\mu_d) \upharpoonright P)$.*

PROOF. Let $\tau \models \mu$. It is easy to see that $\tau_d \models (\Box_{[0,d)} \bigwedge_{p \in P} \neg p) \wedge (\Box_{[0,d)} \neg c) \wedge (\Box_{[d,\infty)} c)$. We can now argue using structural induction on the formula $tr(\mu)$ to show that if $\tau, t \models \mu$ then $\tau_d, t+d \models tr(\mu)$ and therefore it follows that $\tau_d \models \mu_d$. Since $\tau = tail_d(\tau_d) \upharpoonright P$ we have $L(\mu) \subseteq tail_d(L(\mu_d) \upharpoonright P)$.

Base case: It is easy to see that if μ is an atomic formula and $\tau, t \models \mu$ then $\tau_d, t+d \models tr(\mu)$.

Induction step: Let $\tau \models \diamondminus_I \psi$. Then there exists a $t' \geq 0$ such that $t - t' \in I$ and $\tau, t' \models \psi$. Then by induction hypothesis it follows that $\tau_d, t'+d \models tr(\psi)$ and since $(t+d)-(t'+d) \in I$ we have $\tau_d, t+d \models \diamondminus_I(tr(\psi))$. Since $t'+d \geq d$ we have that $\tau_d, t'+d \models c$. So we have $\tau_d, t+d \models \diamondminus_I(c \wedge tr(\psi))$ and therefore $\tau_d, t+d \models tr(\diamondminus_I \psi)$. Similarly we can argue for other cases that if $\tau, t \models \mu$ then $\tau_d, t+d \models tr(\mu)$.

For the converse direction let $t \geq 0$ and $\tau', t+d \models \mu_d$. We now argue that $tail_d(\tau') \upharpoonright P, t \models \mu$. Then it follows that $\tau' \models \mu_d$ implies $tail_d(\tau'), t \models \mu$. The proof is similar to the one given above which uses structural induction on the formula μ to show that if $\tau', t+d \models tr(\mu)$ then $\tau, t \models \mu$. Thus we have $tail_d(L(\mu_d) \upharpoonright P) \subseteq L(\mu)$. ∎

Returning now to our original formula $\varphi \in nr\text{-LTL}^c(\Diamond, \diamondminus)$ which uses the past formulas $\diamondminus_{I_0} p_0, \ldots, \diamondminus_{I_n} p_n$, we first go over to the formula $\varphi_{d_{max}}$ which is satisfied precisely by the models of φ shifted by d_{max} time units. Let $\varphi' = \varphi_{d_{max}}[p'_j/(c \wedge p_j)] \wedge (p'_j \Leftrightarrow (c \wedge p_j))$ be the flattened version of $\varphi_{d_{max}}$. Now define

$$\widehat{\varphi} = \varphi'[q_j/\diamondminus_{I_j} p'_j] \wedge (\nu[p'_j/p_j] \wedge (\bigwedge_{j=0}^{j=n} \Box_{[0,d_{max})} \neg q_j)).$$

Then by the Corollary 6 and Lemma 7 we have:

Theorem 8. *Let $\varphi \in \mathrm{LTL}^c(\Diamond, \diamondminus)$ over the set of propositions P. Then we can construct a formula $\widehat{\varphi} \in \mathrm{LTL}^c(\Diamond)$ over an extended set of propositions P' such that $L(\varphi) = tail_{d_{max}}(L(\widehat{\varphi}) \upharpoonright P)$.* ∎

References

[1] R. Alur, T. Feder, and T. A. Henzinger. The benefits of relaxing punctuality. *J. ACM*, 43(1):116–146, 1996.

[2] P. Bouyer, F. Chevalier, and N. Markey. On the expressiveness of TPTL and MTL. In Ramanujam and Sen [15], pages 432–443.

[3] J.R. Büchi. On a decision method in restricted second order arithmetic. In *Z. Math. Logik Grundlag. Math.*, pages 66–92, 1960.

[4] F. Chevalier, D. D'Souza, and P. Prabhakar. On continuous timed automata with input-determined guards. In S. Arun-Kumar and N. Garg, editors, *FSTTCS*, volume 4337 of *Lecture Notes in Computer Science*, pages 369–380. Springer, 2006.

[5] D. D'Souza, Raj Mohan M., and P. Prabhakar. Flattening Metric Temporal Logic. Technical Report IISc-CSA-TR-2006-11, Indian Institute of Science, Bangalore 560012, India, September 2006. URL: `http://archive.csa.iisc.ernet.in/TR/2006/11/`.

[6] D. D'Souza and M. R. Mohan. Eventual Timed Automata. In Ramanujam and Sen [15], pages 322–334.

[7] D. M. Gabbay, A. Pnueli, S. Shelah, and J. Stavi. On the Temporal Basis of Fairness. In *POPL*, pages 163–173, 1980.

[8] T. A. Henzinger, J. -F. Raskin, and P. -Y. Schobbens. The Regular Real-Time Languages. In Kim Guldstrand Larsen, Sven Skyum, and Glynn Winskel, editors, *ICALP*, volume 1443 of *Lecture Notes in Computer Science*, pages 580–591. Springer, 1998.

[9] Y. Hirshfeld and A. M. Rabinovich. Logics for Real Time: Decidability and Complexity. *Fundam. Inform.*, 62(1):1–28, 2004.

[10] J. A. W. Kamp. *Tense Logic and the Theory of Linear Order*. PhD thesis, University of California, Los Angeles, California, 1968.

[11] R. Koymans. Specifying Real-Time Properties with Metric Temporal Logic. *Real-Time Systems*, 2(4):255–299, 1990.

[12] J. Ouaknine and J. Worrell. On Metric Temporal Logic and Faulty Turing Machines. In Luca Aceto and Anna Ingólfsdóttir, editors, *FoSSaCS*, volume 3921 of *Lecture Notes in Computer Science*, pages 217–230. Springer, 2006.

[13] A. Pnueli. The Temporal Logic of Programs. In *FOCS*, pages 46–57. IEEE, 1977.

[14] P. Prabhakar and D. D'Souza. On the Expressiveness of MTL with Past Operators. In Eugene Asarin and Patricia Bouyer, editors, *FORMATS*, volume 4202 of *Lecture Notes in Computer Science*, pages 322–336. Springer, 2006.

[15] R. Ramanujam and S. Sen, editors. *FSTTCS 2005: Foundations of Software Technology and Theoretical Computer Science, 25th International Conference, Hyderabad, India, December 15-18, 2005, Proceedings*, volume 3821 of *Lecture Notes in Computer Science*. Springer, 2005.