

An Algebraic Characterization of Elementary Equivalence
Fraïssé-Ehrenfeucht Games

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Notation

- \mathfrak{A} denote a structure i.e. a domain and an interpretation.
 $\mathfrak{A} := (A, \mathfrak{a})$
 - A is the *domain/universe* of \mathfrak{A} .
 - $\mathfrak{a} = (R, f, c)$.
- $\mathfrak{A} \models \phi[a_0, \dots, a_{r-1}]$ denotes \mathfrak{A} satisfies ϕ under the assignment $(v_0, \dots, v_{r-1}) \mapsto (a_0, \dots, a_{r-1})$.

Introduction

- **Elementary Equivalence** : Two Structures \mathfrak{A} and \mathfrak{B} are called *elementarily equivalent* (written: $\mathfrak{A} \equiv \mathfrak{B}$) if for every sentence ϕ we have $\mathfrak{A} \models \phi$ iff $\mathfrak{B} \models \phi$.
- **Theory** : $T \in L_0^S$ is said to be a *theory* if T is satisfiable and if it is closed under consequence.
- For every S-structure \mathfrak{A} let $\text{Th}(\mathfrak{A}) := \{\phi \in L_0^S \mid \mathfrak{A} \models \phi\}$. $\text{Th}(\mathfrak{A})$ is called the (first-order) *theory* of \mathfrak{A} .
- For $\Phi \in L_0^S$ let $\Phi \models := \{\phi \in L_0^S \mid \Phi \models \phi\}$
- **Lemma** : For two structures \mathfrak{A} and \mathfrak{B} ,
 $\mathfrak{A} \equiv \mathfrak{B}$ iff $\mathfrak{B} \models \text{Th}(\mathfrak{A})$.

Examples of Theories

1. $\Phi^{\models} := \{\varphi \in L_0^S \mid \models \varphi\}$
2. For $S = S_{gr}$: group theory $Th_{gr} := \Phi_{gr}^{\models}$
3. For $S = \{\in\}$: ZFC set theory $Th_{ZFC} := ZFC^{\models}$
4. For $S = S_{ar}$: (first order) Peano arithmetic $Th_{PA} := \Phi_{PA}^{\models}$

Isomorphisms

- **Partial Isomorphism :** Let \mathfrak{A} and \mathfrak{B} be structures and let p be a map. We call p a *partial isomorphism* from \mathfrak{A} to \mathfrak{B} if and only if $\text{dom}(p) \subset A$, $\text{rg}(p) \subset B$, and p has the following properties:
 - p is injective.
 - p is homomorphic in the following sense:
 - For n-ary $P \in S$ and $\{a_1, \dots, a_n\} \in \text{dom}(p)$,
 $P^{\mathfrak{A}} a_1, \dots, a_n$ iff $P^{\mathfrak{B}} p(a_1), \dots, p(a_n)$
 - For n-ary $f \in S$ and $\{a_1, \dots, a_n, a\} \in \text{dom}(p)$,
 $f^{\mathfrak{A}} a_1, \dots, a_n = a$ iff $P^{\mathfrak{B}} p(a_1), \dots, p(a_n) = p(a)$.
 - For $c \in S$ and $a \in \text{dom}(p)$,
 $c^{\mathfrak{A}} = a$ iff $c^{\mathfrak{B}} = p(a)$.

Isomorphisms

- The empty map i.e. the map with the empty domain is a partial isomorphism from \mathfrak{A} to \mathfrak{B} .
- If S is relational, that is if S contains only relation symbols, then for a_0, \dots, a_{r-1} and b_0, \dots, b_{r-1} the following sentences are equivalent:
 - By setting $p(a_i) := b_i$ for $i < r$ a partial isomorphism from \mathfrak{A} to \mathfrak{B} is determined.
 - For every atomic formula $\phi \in \mathcal{S}_r$,
 $\mathfrak{A} \models \phi[a_0, \dots, a_{r-1}]$ iff $\mathfrak{B} \models \phi[b_0, \dots, b_{r-1}]$.
- Consider $p : \{(2,2), (3,6)\}$. This is a partial isomorphism from $(\mathbb{R}, +, 0)$ to $(\mathbb{Z}, +, 0)$. The equivalence is not true if S contains function symbols and constants.
 $\neg(\mathbb{R}, +, 0) \models v_0 + (v_0 + v_0) \equiv v_1[2, 3],$
 $(\mathbb{Z}, +, 0) \models v_0 + (v_0 + v_0) \equiv v_1[p(2), p(3)]$

Extension

- The equivalence is not even true if we introduce quantifiers.
Ex: $q_0 = \{(2, 3), (3, 4)\}$. Then
 $(\mathbb{R}, <) \models \exists v_2 (v_0 < v_2 \wedge v_2 < v_1)[2, 3]$
 $\neg(\mathbb{Z}, <) \models \exists v_2 (v_0 < v_2 \wedge v_2 < v_1)[q(2), q(3)]$.
- We can see that we cannot extend p , to another partial isomorphism which has an element between 2 and 3 in its domain.
- **Hence, we see that the truth of formulas with quantifiers is preserved under partial isomorphism, provided that they admit certain extensions. This is the main idea.**

Isomorphisms

- **Definition :** \mathfrak{A} and \mathfrak{B} are *finitely isomorphic* (written $\mathfrak{A} \cong_f \mathfrak{B}$) iff there is a sequence $(I_n)_{n \in \mathbb{N}}$ with the following properties:
 1. Every I_n is a non-empty set of partial isomorphisms from \mathfrak{A} to \mathfrak{B} .
 2. (*Forth Property*) For every $p \in I_{n+1}$ and $a \in A$ there is a $q \in I_n$ such that $p \subset q$ and $a \in \text{dom}(q)$
 3. (*Back Property*) For every $p \in I_{n+1}$ and $b \in B$ there is a $q \in I_n$ such that $p \subset q$ and $b \in \text{rg}(q)$
- This means that the partial isomorphisms in I_n can be extended n times.

Isomorphisms

- **Definition :** \mathfrak{A} and \mathfrak{B} are *partially isomorphic* (written $\mathfrak{A} \cong_p \mathfrak{B}$) iff there is a set I with the following properties:
 1. Every I is a non-empty set of partial isomorphisms from \mathfrak{A} to \mathfrak{B} .
 2. (*Forth Property*) For every $p \in I$ and $a \in A$ there is a $q \in I$ such that $p \subset q$ and $a \in \text{dom}(q)$
 3. (*Back Property*) For every $p \in I$ and $b \in B$ there is a $q \in I$ such that $p \subset q$ and $b \in \text{rg}(q)$

Consequences

- **Lemma :**

- If $\mathfrak{A} \cong \mathfrak{B}$ then $\mathfrak{A} \cong_p \mathfrak{B}$
- If $\mathfrak{A} \cong_p \mathfrak{B}$ then $\mathfrak{A} \cong_f \mathfrak{B}$
- If $\mathfrak{A} \cong_f \mathfrak{B}$ and A is finite then $\mathfrak{A} \cong \mathfrak{B}$
- If $\mathfrak{A} \cong_p \mathfrak{B}$ and A is countable then $\mathfrak{A} \cong \mathfrak{B}$

- **Theorem :** Any two countable dense orderings without endpoints are isomorphic.

This is just a special version of the last part. It follows from

- **Lemma :** If $\mathfrak{A} = (A, <^A)$ and $\mathfrak{B} = (B, <^B)$ are dense orderings, then $I : \mathfrak{A} \cong_p \mathfrak{B}$ for $I = \{p \mid p \in Part(\mathfrak{A}, \mathfrak{B}), dom(p) \text{ is finite} \}$

Natural Numbers

- $\mathfrak{N} = (\mathbb{N}, +, *, 0, 1)$.
 \mathfrak{N} satisfies the Peano axiom system :
 - (P1) $\forall x \neg \sigma(x) \equiv 0$
 - (P2) $\forall x \forall y (\sigma(x) \equiv \sigma(y) \rightarrow x \equiv y)$
 - (P3) $\forall X ((X0 \wedge \forall x (Xx \rightarrow X\sigma x)) \rightarrow \forall y Xy)$
- Consider the following “successor axioms” Φ_σ :
 - $\forall x (\neg x \equiv 0 \leftrightarrow \exists y \sigma y \equiv x)$
 - $\forall x \forall y (\sigma(x) \equiv \sigma(y) \rightarrow x \equiv y)$
 - $\forall m \geq 1 : \forall x \neg \underbrace{\sigma \dots \sigma}_m x \equiv x$
- \mathfrak{N} is a model for Φ_σ .
- Any two models for Φ_σ are finitely isomorphic.

Ehrenfeucht Games

- There are two structures A and B, and two players who are commonly called Spoiler and Duplicator.
- Each step in the game consists of a move of Spoiler, followed by a move of Duplicator. Spoiler chooses an element of one of the two structures, and Duplicator must then choose an element of the other structure.
- So after n steps, two sequences have been chosen, one from A and one from B: $(a_0, \dots, a_{n-1}; b_0, \dots, b_{n-1})$
- The game is a win for Spoiler if $(a_0, \dots, a_{n-1}; b_0, \dots, b_{n-1})$ is not a partial isomorphism from A to B.
- **Ehrenfeucht's Theorem**
For any two structures \mathfrak{A} and \mathfrak{B} :
 $\mathfrak{A} \equiv \mathfrak{B}$ iff Duplicator has a winning strategy for $G(\mathfrak{A}, \mathfrak{B})$

Applications

The theory behind back-and-forth games uses very few assumptions about the logic in question. As a result, these games are one of the few model-theoretic techniques that apply as well to finite structures as they do to infinite ones, and this makes them one of the cornerstones of theoretical computer science.

Applications

To measure the expressive strength of formal languages, for example database query languages. A typical result might say, for example, that a certain language can't distinguish between "even" and "odd"; we would prove this by finding, for each level n of complexity of formulas of the language, a pair of finite structures for which Duplicator has a winning strategy in the back-and-forth game of level n , but one of the structures has an even number of elements and the other has an odd number.

Applications

Let Z be a binary relation which relates states of A to states of B . Then we call Z a bisimulation between A and B if Duplicator can use Z as a nondeterministic winning strategy in the back-and-forth game between A and B where the first pair of moves of the two players is to choose their starting states.

Fraïssé's Theorem

Let \mathfrak{A} and \mathfrak{B} be structures. Then

$\mathfrak{A} \equiv \mathfrak{B}$ iff $\mathfrak{A} \cong_f \mathfrak{B}$

Successor Axioms

Applications of Fraïssé's Theorem

- **Lemma :** Any two dense orderings are elementarily equivalent. In particular, $(\mathbb{R}, <) \equiv (\mathbb{Q}, <)$.
- **Lemma :** Any two $(\sigma, 0)$ structures satisfying Φ_σ are elementarily equivalent.
- **Lemma :** For a theory $T \subset L_0^S$ the following are equivalent :
 - T is complete i.e. $\forall \phi$ either $\phi \in T$ or $\neg \phi \in T$
 - Any two models of T are elementarily equivalent.
- **Lemma :** The theory of dense orderings is complete and decidable.
- **Lemma :** The theory Φ_σ^{\models} of successor structures is complete and decidable.

Thank You

References

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