More undecidable problems

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20 November 2012
1 More problems about Turing Machines
Problem (a)

Is it decidable whether a given Turing machine has at least 481 states? Assume that the TM is given using the encoding below:

\[ 0^n10^m10^k10^s10^t10^u10^v10^p10^q10^b1010^p’10^a’10^q’10^b’100\cdots10^p''10^a''10^q''10^b''10. \]

00010000100101001000100010000 1000101000100 1000100100100 1010101010.
Problem (a)

Is it decidable whether a given Turing machine has at least 481 states? Assume that the TM is given using the encoding below:

\[0^n10^m10^k10^s10^t10^r10^u10^v10^p10^q10^b1010^p'10^q'10^b'100\ldots10^p''10^q''10^b''10.\]

Yes, it is.

We can give a TM \(N\) which given \(\text{enc}(M)\)

- Counts the number of states in \(M\) upto 481.
- Accepts if it reaches 481, rejects otherwise.
More problems about Turing Machines

More decidable/undecidable problems

Problem (b)

Is it decidable whether a given Turing machine takes more than 481 steps on input $\epsilon$ without halting?

00010000100101001000100010000 1 01000101000100 1 0100100100100 1 010101010.

Yes, it is. We can give a TM $N$ which given $\text{enc}(M)$

Uses 4 tapes: On the 4th tape it writes 481 0's.

Uses the first 3 tapes to simulate $M$ on input $\epsilon$, like the universal TM $U$.

Blanks out a 0 from 4th tape for each 1-step simulation done by $U$.

Rejects if $M$ halts before all 0's are blanked out on 4th tape, accepts otherwise.
Problem (b)

Is it decidable whether a given Turing machine takes more than 481 steps on input $\epsilon$ without halting?

Yes, it is. We can give a TM N which given $\text{enc}(M)$

- Uses 4 tapes: On the 4th tape it writes 481 0’s.
- Uses the first 3 tapes to simulate $M$ on input $\epsilon$, like the universal TM $U$.
- Blanks out a 0 from 4th tape for each 1-step simulation done by $U$.
- Rejects if $M$ halts before all 0’s are blanked out on 4th tape, accepts otherwise.
Problem (c)

Is it decidable whether a given Turing machine takes more than 481 steps on some input without halting?

00010000100101001000100010000 1 01000101000100 1 0100100100100 1 010101010.
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Problem (c)
Is it decidable whether a given Turing machine takes more than 481 steps on some input without halting?

Yes, it is.
Check if $M$ runs for more than 481 steps on each input $x$ of length upto 481. If so accept, else reject.
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Problem (d)

Is it decidable whether a given Turing machine takes more than 481 steps on all inputs without halting?

00010000100101001000100010000 1 01000101000100 1 0100100100100 1 010101010 00010000101001001000100010000 1 01000101000100 1 0100100100100 1 010101010.
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Problem (d)
Is it decidable whether a given Turing machine takes more than 481 steps on all inputs without halting?

```
00010000100101001000100010000 1 01000101000100 1 0100100100100 1 010101010
```

Yes, it is.
Check if $M$ runs for more than 481 steps on each input $x$ of length upto 481. If so accept, else reject.

```
1 2 3 ··· 481 482
 detach a a b a b a a a b b ···
```
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Problem (e)
Is it decidable whether a given Turing machine moves its head more than 481 cells away from the left-end marker, on input $\epsilon$?

000100001010010010000100100010000 1 01000101000100 1 0100100100100 1 010101010.

Yes, it is. Simulate $M$ on $\epsilon$ for up to $m \cdot 482 \cdot k$ steps. If $M$ visits the 482nd cell, accept, else reject.
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Problem (e)

Is it decidable whether a given Turing machine moves its head more than 481 cells away from the left-end marker, on input $\epsilon$?

Yes, it is.

Simulate $M$ on $\epsilon$ for upto $m^{481} \cdot 482 \cdot k$ steps. If $M$ visits the 482nd cell, accept, else reject.
Problem (f)

Is it decidable whether a given Turing machine accepts the null-string $\epsilon$?
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More decidable/undecidable problems

Problem (f)

Is it decidable whether a given Turing machine accepts the null-string $\epsilon$?

No.
If it were decidable, say by a TM $N$, then we could use $N$ to decide HP as follows: Define a new machine $N'$ which given input $M\#x$, outputs the description of a machine $M'$ which:

- erases its input
- writes $x$ on its input tape
- Behaves like $M$ on $x$
- Accepts if $M$ halts on $x$.

$N'$ then calls $N$ with input $M'$. $N$ accepts $M'$ iff $M'$ accepts $\epsilon$ iff $M$ halts on $x$. 
Turing machine $M'$ for Problem (f)

$L(M') = \begin{cases} 
A^* & \text{if } M \text{ halts on } x \\
\emptyset & \text{if } M \text{ does not halt on } x.
\end{cases}$
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Problem (g)

Is it decidable whether a given Turing machine accepts any string at all? That is, is \( L(M) \neq \emptyset \)?
Problem (h)

Is it decidable whether a given Turing machine accepts all strings? That is, is $L(M) = A^*$?
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More decidable/undecidable problems

Problem (i)
Is it decidable whether a given Turing machine accepts a finite set?
Problem (j)

Is it decidable whether a given Turing machine accepts a regular set?

Given $M$ and $x$, build a new machine $M'$ that behaves as follows:

1. Saves its input $y$ on tape 2.
2. Writes $x$ on tape 1.
3. Runs as $M$ on $x$.
4. If $M$ gets into a halting state, then $M'$ takes back control, runs as $M_R$ on $y$, (Here $M_R$ is any TM that accepts a non-regular language $R$, say $R = \{a^n b^n | n \geq 0\}$).

$M'$ accepts iff $M_R$ accepts.
Problem (j)

Is it decidable whether a given Turing machine accepts a regular set?

Given $M$ and $x$, build a new machine $M'$ that behaves as follows:

1. Saves its input $y$ on tape 2.
2. Writes $x$ on tape 1.
3. Runs as $M$ on $x$.
4. If $M$ gets into a halting state, then
   - $M'$ takes back control,
   - Runs as $M_R$ on $y$,
   - (Here $M_R$ is any TM that accepts a non-regular language $R$, say $R = \{a^n b^n \mid n \geq 0\}$).
   - $M'$ accepts iff $M_R$ accepts.
Turing machine $M'$ for Problem (j)

$$L(M') = \begin{cases} R & \text{if } M \text{ halts on } x \\
\emptyset & \text{if } M \text{ does not halt on } x. \end{cases}$$
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Problem (k)

Is it decidable whether a given Turing machine accepts a CFL?
More decidable/undecidable problems

Problem (I)
Is it decidable whether a given Turing machine accepts a recursive set?