Closure under boolean ops

Induction

NFA’s

Closure properties of regular languages

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Outline

1. Closure under boolean ops
2. Induction
3. NFA’s
Closure properties

- Class of Regular languages is closed under
  - Complement, intersection, and union.
  - Concatenation, Kleene iteration.
- Non-deterministic Finite-state Automata (NFA) = DFA.
Closure under complementation

- **Idea:** Flip final states.
- **Formal construction:**
  - Let $A = (Q, s, \delta, F)$ be a DFA over alphabet $A$.
  - Define $B = (Q, s, \delta, Q - F)$.
  - Claim: $L(B) = A^* - L(A)$.

**Proof of claim**

- $L(B) \subseteq A^* - L(A)$.
  - $w \in L(B) \implies \hat{\delta}(s, w) \in (Q - F)$.
  - $\implies \hat{\delta}(s, w) \not\in F$
  - $\implies w \not\in L(A)$
  - $\implies w \in A^* - L(A)$.

- $L(B) \supseteq A^* - L(A)$. 
Closure under boolean ops

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Closure under intersection

Product construction. Given DFA’s $\mathcal{A} = (Q, s, \delta, F)$, $\mathcal{B} = (Q', s', \delta', F')$, define product $\mathcal{C}$ of $\mathcal{A}$ and $\mathcal{B}$:

$$\mathcal{C} = (Q \times Q', (s, s'), \delta'', F \times F'),$$

where $\delta''( (p, p'), a) = (\delta(p, a), \delta'(p', a))$.

Product construction example

![Diagrams of A, B, and A × B with transitions](image)
Correctness of product construction

Claim: \( L(C) = L(A) \cap L(B) \).

Proof of claim \( L(C) = L(A) \cap L(B) \).

- \( L(C) \subseteq L(A) \cap L(B) \).
  \[
  w \in L(C) \implies \hat{\delta}''((s, s'), w) \in F \times F'.
  \]
  \[
  \implies (\hat{\delta}(s, w), \hat{\delta}'(s', w)) \in F \times F' \text{ (by subclaim)}
  \]
  \[
  \implies \hat{\delta}(s, w) \in F \text{ and } \hat{\delta}'(s', w) \in F'
  \]
  \[
  \implies w \in L(A) \text{ and } w \in L(B)
  \]
  \[
  \implies w \in L(A) \cap L(B).
  \]

- \( L(C) \supseteq L(A) \cap L(B) \).
  \[
  \text{Subclaim: } \hat{\delta}''((s, s'), w) = (\hat{\delta}(s, w), \hat{\delta}'(s', w)).
  \]
Closure under union

Follows from closure under complement and intersection since

\[ L_1 \cup L_2 = \overline{L_1 \cap L_2}. \]
Closure under union

- Follows from closure under complement and intersection since
  \[ L_1 \cup L_2 = \overline{L_1} \cap \overline{L_2}. \]
- Can also do directly by product construction: Given DFA’s
  \( \mathcal{A} = (Q, s, \delta, F) \), \( \mathcal{B} = (Q', s', \delta', F') \), define \( \mathcal{C} \):
  \[ \mathcal{C} = (Q \times Q', (s, s'), \delta'', (F \times Q') \cup (Q \times F')) \], where
  \[ \delta''((p, p'), a) = (\delta(p, a), \delta(p', a)). \]

Union construction
Principle of Mathematical Induction

- $\mathbb{N} = \{0, 1, 2 \ldots\}$
- $P(n)$: A statement $P$ about a natural number $n$.
- Example:
  - $P(n) = \text{“}n \text{ is even.”}$
  - $P_1(n) = \text{“}\text{Sum of the numbers } 1 \ldots n \text{ equals } n(n + 1)/2.\text{”}$
  - $P_2(n) = \text{“}\text{For all } w \in A^*, \text{ if length of } w \text{ is } n \text{ then } \hat{\delta}''((s, s'), w) = (\hat{\delta}(s, w), \hat{\delta}'(s', w)).\text{”}$

Principle of Induction

If a statement $P$ about natural numbers

- is true for 0 (i.e. $P(0)$ is true), and,
- is true for $n + 1$ whenever it is true for $n$ (i.e. $P(n) \implies P(n + 1)$)

then $P$ is true of all natural numbers (i.e. “For all $n$, $P(n)$” is true).
Exercise: Prove the Subclaim:

\[ \hat{\delta}''((s, s'), w) = (\hat{\delta}(s, w), \hat{\delta}'(s', w)). \]

using induction.
Nondeterministic Finite-state Automata (NFA)

- Allows multiple start states.
- Allows more than one transition from a state on a given letter.

A word is accepted if there is some path on it from a start to a final state.
Example NFA's

NFA for “contains $abb$ as a subword”
NFA definition

- Mathematical representation of NFA
  \[ A = (Q, S, \Delta, F), \text{ where } S \subseteq Q, \text{ and } \Delta : Q \times A \to 2^Q. \]
  - Define relation \( p \xrightarrow{w} q \) which says there is a path from state \( p \) to state \( q \) labelled \( w \).
    - \( p \xrightarrow{\epsilon} p \)
    - \( p \xrightarrow{ua} q \) iff there exists \( r \in Q \) such that \( p \xrightarrow{u} r \) and \( q \in \Delta(r, a) \).
  - Define \( L(A) = \{ w \in A^* \mid \exists s \in S, f \in F : s \xrightarrow{w} f \} \).

- NFA \( \rightarrow \) DFA: Subset construction
  - Example: determinize NFA for “contains \( abb \).”
  - Formal construction
  - Correctness
Closure under concatenation and Kleene iteration

- Concatenation of languages:
  \[ L \cdot M = \{ u \cdot v \mid u \in L, \ v \in M \}. \]

- Kleene iteration of a language:
  \[ L^* = \{ \epsilon \} \cup L \cup L^2 \cup L^3 \cup \cdots, \]
  where
  \[ L^n = L \cdot L \cdots L \ (n \text{ times}). \]
  \[ = \{ w_1 \cdots w_n \mid \text{each } w_i \in L \}. \]