1. This question shows that the pumping lemma for regular languages is not a sufficient condition for regularity. Consider the language $L$ below

$$(a + b + c)^* cc (a + b + c)^* + \bigcup_{n \geq 0} (a^+ c)^n (b^+ c)^n.$$ 

(a) Show that $L$ is not regular.
(b) Show that $L$ nonetheless satisfies the conditions of the pumping lemma. That is, the adversary has a winning strategy in the game over $L$. In other words, show that there exists an $n > 0$ such that for any string $xyz$ in $L$, $|y| \geq n$, there exists strings $u, v, w$ such that $y = uvw$, $v \neq \epsilon$, and for all $i$, the string $xuv^iwz \in L$.

2. We saw in class that ultimate periodicity of lengths of strings in a language was a necessary condition for regularity of the language. Argue that over a single-letter alphabet, the condition of ultimate periodicity is also sufficient for regularity.

3. Show that the following definitions of ultimate periodicity for a set of natural numbers $X$ are equivalent:

(a) There exist nonzero natural numbers $n$ and $p$ such that for all $m \geq n$, $m \in X$ iff $m + p \in X$.
(b) There exist nonzero natural numbers $n$ and $p$ such that for all $m \geq n$, $m \in X$ implies $m + p \in X$.

Use the second definition to give a simple solution to the problem given in the assignment: If $L$ is a subset of $\{a\}^*$, then $L^*$ is regular.

4. Construct a language $L \subseteq \{a, b\}^*$ with the property that neither $L$ nor its complement contains an infinite regular set.

5. For a set of natural numbers $A$, define $\text{binary}(A)$ to be the set of binary representations of numbers in $A$. Similarly define $\text{unary}(A)$ to be the set of “unary” representations of numbers in $A$: $\text{unary}(A) = \{1^n \mid n \in A\}$. Thus for $A = \{2, 3, 6\}$, $\text{binary}(A) = \{10, 11, 110\}$ and $\text{unary}(A) = \{11, 111, 111111\}$.

Consider the two propositions below:

(a) For all $A$, if $\text{binary}(A)$ is regular then so is $\text{unary}(A)$.
(b) For all $A$, if $\text{unary}(A)$ is regular then so is $\text{binary}(A)$. 
One of the statements above is true and the other is false. Which is true and which is false?

6. Describe the equivalence classes of the canonical Myhill-Nerode relation for the language of equal number of a’s and b’s over the alphabet \{a, b\}. Draw a representation of the resulting canonical deterministic automaton for the language.

7. Give a regular expression which describes the language defined by the MSO sentence below over the alphabet \{a, b\}.

\[ \forall x \forall y ((Q_b(x) \land \text{succ}(x, y)) \implies Q_a(y)) \land \\
\exists X ((\exists x (\text{zero}(x) \land x \in X)) \land \\
(\exists x (\text{last}(x) \land \neg x \in X)) \land \\
(\forall x \forall y (\text{succ}(x, y) \implies (x \in X \Leftrightarrow \neg y \in X)) \land \\
\forall x (x \in X \implies Q_b(x))). \]

8. Give an MSO sentence describing the language \((ab^*a)^*\). Try to give a sentence as simple (logically understandable) as possible. In particular don’t follow the route of constructing an automaton and then converting it to a sentence.

9. Let \(L\) be a regular language. Show that the language

\[ \{ x \mid \exists y : |y| = 2^{|x|}, \text{ and } xy \in L \}. \]

is also regular. Give an explicit construction.

10. Professor Jones proposes the following procedure for finding the canonical automaton for a language \(L\) starting from a DFA \(A\) for \(L\).

(a) Reverse the transitions in \(A\), make the initial state final, and the final states initial, to obtain an NFA \(B\) for \(\text{rev}(L)\).

(b) Determinize \(B\) using the subset construction to get \(C\).

(c) Repeat the above two steps, starting from \(C\), to get respectively an NFA \(D\), and DFA \(E\).

(d) Return \(E\) as the canonical automaton for \(A\).

Is Jones’ procedure correct? Justify your answer.

Questions 9 and 10 may be discussed with your classmates or TA’s. Hand in written answers to all questions.