Reductions and Rice’s theorems

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Outline

1. Reductions

2. Rice’s theorems
Let $L \subseteq A^*$ and $M \subseteq B^*$ be two languages. We say $L$ reduces to $M$ and write $L \leq M$ iff there exists a computable map $\sigma : A^* \rightarrow B^*$ such that

$$w \in L \text{ iff } \sigma(w) \in M.$$
Reductions and recursive/re-ness

Theorem
If $L \leq M$ then:
1. If $M$ is r.e. then so is $L$.
2. If $M$ is recursive then so is $L$.

Or to put it differently:

Theorem
If $L \leq M$ then:
1. If $L$ is not r.e. then neither is $M$.
2. If $L$ is not recursive then neither is $M$.
Examples of reductions

Let $L$ be the language $\{M \mid M$ accepts $\epsilon\}$. Then

$$\text{HP} \leq L.$$ 

Describe a computable map $\sigma$ which witnesses the reduction. Hence, since HP is undecidable (i.e. not recursive) so is $L$. 
Let $L$ be the language $\{M \mid M$ accepts a regular language\}. Then

$$\neg\text{HP} \leq L.$$

- Describe a computable map $\sigma$ which witnesses the reduction.
- Hence, since $\neg\text{HP}$ is undecidable (i.e. not recursive) so is $L$.
- In fact, since $\neg\text{HP}$ is not r.e., we can say that $L$ is not r.e..
Rice’s theorem

Theorem (Rice)

Any non-trivial property of r.e. languages is undecidable.
Rice’s theorem

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Any non-trivial property of r.e. languages is undecidable.

Theorem (Rice)
Any non-monotone property of r.e. languages is not even recursively enumerable.
A property $P$ of languages over an alphabet $A$ is a subset of languages over $A$. 
A property $P$ is a non-trivial property of r.e. languages, if there is at least one r.e. language $L$ satisfying $P$, and another $L'$ not satisfying $P$. 

E.g. "is empty" is non-trivial; "is not accepted by a TM" is trivial.

A property $P$ of languages is monotone (w.r.t r.e. languages) if for all r.e. sets $A$ and $B$, whenever $A \subseteq B$ and $P(A)$, we have $P(B)$. I.e., $P$ is monotone if whenever a set has the property, then all supersets of that set have it as well. "is infinite" is monotone, "$L(M)$ is finite" is not monotone.
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E.g. “is empty” is non-trivial.
A property $P$ is a **non-trivial** property of r.e. languages, if there is at least one r.e. language $L$ satisfying $P$, and another $L'$ not satisfying $P$.

- E.g. “is empty” is non-trivial
- “is not accepted by a TM” is trivial.

A property $P$ of languages is **monotone** (w.r.t r.e. languages) if for all r.e. sets $A$ and $B$, whenever $A \subseteq B$ and $P(A)$, we have $P(B)$.

IOW, $P$ is monotone if whenever a set has the property, then all supersets of that set have it as well.
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- “is infinite” is monotone,
- “$L(M)$ is finite” is not monotone.
Rice’s theorems

For a property $P$, we define

$$L_P = \{ M \mid L(M) \text{ satisfies } P \}.$$  

**Theorem (Rice)**

Any non-trivial property of r.e. languages is undecidable. That is, if $P$ is a non-trivial property of r.e. languages, then the language $L_P$ is not recursive.

**Theorem (Rice)**

Any non-monotone property of r.e. languages is not even recursively enumerable. That is, if $P$ is a non-monotone property of r.e. languages, then the language $L_P$ is not even recursively enumerable.
Proof of Rice’s Theorem 1

- Let $P$ be a non-trivial property of r.e. languages. Then there are TM’s $K$ and $T$ such $L(K)$ satisfies $P$ and $L(T)$ does not satisfy $P$.
- We show that $L_P = \{ M | L(M) \text{ satisfies } P \}$ is not recursive.
- Case 1: If $\emptyset$ satisfies $P$. We reduce HP to $L_P$.
- Given $M \# x$, construct a machine $M' = \sigma(M \# x)$ that on input $y$
  - saves $y$ on a separate track
  - writes $x$ on its tape
  - runs as $M$ on input $x$
  - if $M$ halts on $x$, $M'$ runs as $K$ on $y$ and accepts iff $K$ accepts.

$$L(M') = \begin{cases} 
  L(K) & \text{if } M \text{ halts on } x \\
  \emptyset & \text{if } M \text{ does not halt on } x.
\end{cases}$$
Proof of Rice’s Theorem 2

- Let $P$ be a non-monotone property of r.e. sets.
- Then there are TM’s $K$ and $T$ such that $L(K) \subseteq L(T)$ and $L(K)$ satisfies $P$ but $L(T)$ does not.
- We show $\neg HP \leq LP$.
- Given $M\#x$ output the description of $M'$ that
  - Given input $y$ on Tape 1.
  - Copies $y$ on Tape 2, writes $x$ on Tape 3
  - Run (in an interleaved fashion) as $M$ on $x$, $K$ on $y$, and $T$ on $y$.
  - accept iff either
    - $K$ accepts $y$, or,
    - $M$ halts on $x$ and $T$ accepts $y$. 
Proof of Rice’s Theorem 2

Notice that:

\[ L(M') = \begin{cases} 
  L(K) & \text{if } M \text{ does not halt on } x \\
  L(T) & \text{if } M \text{ halts on } x.
\end{cases} \]