Undecidability of the Halting Problem

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Outline

1. Universal Turing machine
2. Halting Problem
3. Some corollaries
We can construct a TM $U$ that takes the encoding of a TM $M$ and its input $x$, and “interprets” $M$ on the input $x$.

$U$ accepts if $M$ accepts $x$, rejects if $M$ rejects $x$, and loops if $M$ loops on $x$. 
Encoding a TM as a \(\{0, 1\}\)-string

\[0^n1^m1^k1^s1^t1^r1^u1^v\ \ 1\ 0^p1^a1^q1^b1^0\ \ 1\ 0^p'1^a'1^q'1^b'1^00\ \ \ldots\ \ 1\ 0^{p''}1^a''1^q''1^b''1^0.\]

represents a TM \(M\) with

- states \(\{1, 2, \ldots, n\}\).
- Tape alphabet \(\{1, 2, \ldots, m\}\).
- Input alphabet \(\{1, 2, \ldots, k\}\) (with \(k < m\)).
- Start state \(s \in \{1, 2, \ldots, n\}\).
- Accept state \(t \in \{1, 2, \ldots, n\}\).
- Reject state \(r \in \{1, 2, \ldots, n\}\).
- Left-end marker symbol \(u \in \{k + 1, \ldots, m\}\).
- Blank symbol \(v \in \{k + 1, \ldots, m\}\).
- Each string \(0^p1^a1^q1^b1^0\) represents the transition \((p, a) \rightarrow (q, b, L)\).
Example encoding of TM and its input

Input is encoded as $0^a10^b10^c$ etc.
What does the following TM do on input 001010?

0001000010010100100010001000010100010010010001001
0100010100010010010101010101010101010.
How the universal Turing machine works

- Use 3 tapes: for input $M \# x$, for current configuration, and for current state and position of head.
- Repeat:
  - Execute the transition of $M$ applicable in the current config.
- Accept if $M$ gets into $t$ state, Reject if $M$ gets into $r$ state.
Fix an encoding $enc$ of TM's as above.

Define the language

$$HP = \{enc(M)\#enc(x) \mid M \text{ halts on } x\}.$$
Undecidability of HP

Theorem (Turing)

The language HP is not recursive.
### Proving undecidability of HP

Assume that we have a Turing machine $M$ which decides HP. Then we can compute the entries of the table below:

|     | $\epsilon$ | 0 | 1 | 00 | 01 | 10 | 11 | 000 | 001 | 010 | 011 | 111 | ... |
|-----|-------------|---|---|----|----|----|----|-----|-----|-----|-----|-----|-----|-----|
| $M_\epsilon$ | L | H | L | L | H | H | L | L | L | L | L | L | ... |
| $M_0$ | L | L | L | L | L | L | L | L | L | L | L | L | ... |
| $M_1$ | H | H | L | H | L | H | L | H | L | H | H | ... |
| $M_{00}$ | L | L | L | L | L | L | L | L | L | L | L | L | ... |
| $M_{01}$ | L | H | L | H | L | H | L | H | L | L | L | L | ... |
| $M_{10}$ | H | H | L | H | L | H | L | L | H | L | H | ... |
| $M_{11}$ | L | H | L | L | L | H | L | L | L | L | L | L | ... |
| $M_{000}$ | L | L | L | L | L | L | H | L | L | L | H | L | ... |
| ... |   |   |   |   |   |   |   |   |   |   |   |   |   |

- For each $x \in \{0, 1\}^*$ let $M_x$ denote the TM
  - $M$, if $x$ is the encoding of TM $M$ with input alphabet $0, 1$.
  - $M_{\text{loop}}$ otherwise, where $M_{\text{loop}}$ is a one-state Turing machine that loops on all its inputs.
A TM $N$ that behaves differently from all TM’s

Let us assume we have a TM $M$ that decides HP. Then we can define a TM $N$ as follows: Given input $x \in \{0, 1\}^*$, it

- runs as $M$ on $x \# x$.
- If $M$ accepts (i.e. $M_x$ halts on $x$), goes to a new “looping” state $l$ and loops there.
- If $M$ rejects (i.e. $M_x$ loops on $x$), goes to the accept state $t'$.

$N$ essentially “complements the diagonal” of the table: Given input $x \in \{0, 1\}^*$ it halts iff $M_x$ loops on $x$.

Consider $y = enc(N)$. Then $y$ cannot occur as any row of the table since the behaviour of $N$ differs from all rows in the table. This is a contradiction.
Complement of HP is not r.e.

Fact 1: If \( L \) and \( \overline{L} \) are both r.e. then \( L \) (and \( \overline{L} \)) must be recursive.

- Let \( M \) accept \( L \) and \( M' \) accept \( \overline{L} \).
- We can construct a total TM that simulates \( M \) and \( M' \) on given input, one step at a time.
- Accept if \( M \) accepts, Reject if \( M' \) accepts.

Fact 2: HP is recursively enumerable.

- Just run the universal TM \( U \) on input \( M \# x \); accept iff \( U \) halts (i.e. \( M \) accepts or rejects \( x \)).

Corollary

The language \( \neg \)HP is not even recursively enumerable.
Universal Turing machine Halting Problem

Some corollaries

Where HP lies

All languages over A

- Regular
- RE
- Recursive
- CFL
- DCFL
- $a^n b^n c^n$
- $a^n b^n$
- $a^n c^n$
- $\neg HP$
- HP

$a^n b^n c^n$