Turan’s Theorem

1 Introduction

Extremal graph theory is the branch of graph theory that studies extremal (maximal or minimal) graphs which satisfy a certain property. Extremality can be taken with respect to different graph invariants, such as order, size or girth. For example, a simple extremal graph theory question is ”which acyclic graphs on \( n \) vertices have the maximum number of edges?” The extremal graphs for this question are trees on \( n \) vertices, which have \( n - 1 \) edges.[3]

Paul Turan, a Hungarian mathematician, worked extensively in the fields of Number Theory and Graph Theory. He had a long collaboration with fellow Hungarian mathematician Paul Erdos, lasting 46 years and resulting in 28 joint papers. Erdos wrote of Turan, ”In 1940-1941 he created the area of extremal problems in graph theory which is now one of the fastest-growing subjects in combinatorics.” The field is known more briefly today as extremal graph theory, which is said to have been founded as a result of the following theorem by Turan.[4]

2 Turan’s Theorem (1941) - Statement

**Theorem 1** Among \( n \)-vertex simple graphs with no \( K_{r+1}, T_{n,r} \) has the maximum number of edges. Here, \( K_{r+1} \) refers to the \((r+1)\)-clique and \( T_{n,r} \) refers to the Turan Graph on \( n \) vertices having \( r \) partitions.

This theorem generalises a previous result by Mantel (1907), which states that ”the maximum number of edges in an \( n \)-vertex triangle-free simple graph is \( \lfloor n^2/4 \rfloor \).” Observe that Mantel’s theorem is a special case of the Turan’s theorem with \( r = 2 \).

3 Motivation: \( k \)-Chromatic Graphs

It might be interesting to know which are the smallest and largest \( k \)-chromatic graphs with \( n \) vertices.
What is the minimum size among $k$-chromatic graphs with $n$ vertices?

**Proposition 1** Every $k$-chromatic graph with $n$ vertices has at-least $\binom{k}{2}$ edges.

**Proof** A $k$-chromatic graph has a $k$-vertex coloring, which can be viewed as a $k$-partition of the vertex set, where each partition is an independent set. Suppose we have a proper $k$-coloring of a $k$-chromatic graph. For any pair of colors in the graph, say $i$ and $j$, there exists at-least one edge with end points of colors $i$ and $j$. If such an edge does not exist, then the vertices of colors $i$ and $j$ could be combined into a single color. As this new coloring would use fewer colors, this would contradict our assumption, that the graph is $k$-chromatic (cannot be colored in fewer than $k$ colors). Since there are $\binom{k}{2}$ distinct pairs of colors, there must be at-least $\binom{k}{2}$ edges. Note that the equality clearly holds for a complete graph on $k$-vertices plus $n - k$ isolated vertices. □

What is the maximum size among $k$-chromatic graphs with $n$ vertices?

Suppose we have a proper $k$-coloring. As long as we can find pairs of non-adjacent vertices having different colors, we can continue to add edges without increasing the chromatic number. Thus, to find the maximum possible edges in $k$-chromatic graphs, we will only consider graphs without such vertex pairs.

**Definition 1** A complete multipartite graph is a simple graph $G$ whose vertices can be partitioned into sets such that two vertices are adjacent if and only if they are not in the same partite sets. Equivalently, every component of $G$ is a complete graph. For $k \geq 2$, the complete $k$-partite graph with partite sets of sizes $n_1, n_2, ..., n_k$ is written as $K_{n_1,n_2,...,n_k}$. ♦

4 Turan Graph

The Turan Graph, denoted $T_{n,r}$ is the complete $r$-partite graph with $n$ vertices, whose partite sets differ in size by at-most 1. By the pigeon-hole principle, every partite set has size either $\lceil n/r \rceil$ or $\lfloor n/r \rfloor$.

**Lemma 2** Among simple $r$-partite graphs (that is, $r$-colorable) with $n$ vertices, the Turan graph $T_{n,r}$ is the unique graph with the most edges.
As discussed above, we can add edges without increasing the chromatic number until it becomes a complete multipartite graph. Now, given a complete $r$-partite graph with partite sets differing by more than 1 in size, we can move a vertex $v$ from the largest partite set (size $i$) to the smallest partite set (size $j$). The edges not involving $v$ remain the same as before, but $v$ gains $i - 1$ neighbours in its old partite set, and loses $j$ neighbours in its new partite set. Since $i - 1 > j$, the number of edges increases due to this switch. Hence, we maximize the number of edges only by equalizing the sizes of all partite sets, as in $T_{n,r}$.

What happens if we wish to add more edges than in $T_{n,r}$? Does it force the chromatic number to be $r + 1$? We have seen (Mantel 1907) that there are graphs with chromatic number 2, that have no triangles. But if we have edges more than $\lfloor n^2/4 \rfloor$ on an $n$-vertex graph, then we are forced not only to use 3 colors, but also to have $K_3$(triangle) as a subgraph. Turan generalised this as follows: For an $r$-colorable graph with $n$ vertices, if we go beyond the maximum no. of edges, then we are forced not only to use $r + 1$ colors, but also to have $K_{r+1}$(i.e $r + 1$-clique) as a subgraph.

5 Turan’s Theorem (proof)

**Theorem 3** Among the $n$-vertex simple graphs with no $r+1$-clique, $T_{n,r}$, has the maximum number of edges.

**Proof** Every $r$-colorable(or $r$-partite) graph, including Turan graph $T_{n,r}$, has no $r + 1$-clique, since each partite set contributes at-most one vertex to each clique. If we can prove that the maximum edges is achieved by an $r$-partite graph, then Lemma 2 implies that the required graph is $T_{n,r}$. Thus, it suffices to prove that for every graph $G$ that has no $r + 1$-clique, there is an $r$-partite graph $H$ with the same vertex set as $G$ i.e $V(H) = V(G)$, and at-least as many edges i.e $e(H) \geq e(G)$. 

1-3
We prove this by induction on \( r \).
For the base case \( r = 1 \), any simple graph with no 2-clique is a null-graph (graph with no edges), and is trivially 1-partite. Thus, in this case, \( H = G \).
For the induction step, \( G \) is an \( n \)-vertex simple graph with no \( r+1 \)-clique, where \( r > 1 \). Let \( x \in V(G) \) be a vertex of degree \( k = \Delta(G) \). Let the sub-graph \( G' \) be the induced sub-graph of \( G \) by the set \( N(x) \), where \( N(x) \) is the set of neighbours of \( x \).

**Claim 4** If \( G \) has no \( r+1 \)-clique, then \( G' \) has no \( r \)-clique.

As \( x \) is adjacent to every vertex in \( G' \), if \( G' \) had an \( r \)-clique then \( G \) would have an \( r+1 \)-clique, which would be a contradiction.

Thus, we can apply the induction hypothesis to \( G' \). Thus, there exists an \( r-1 \)-partite graph \( H' \) with \( V(H') = V(G') = N(x) \) and \( e(H') \geq e(G') \). Note that \( V(H') = N(x) = k \). Let \( H \) be the graph formed from \( H' \) by joining all of \( N(x) \) to all of \( S = V(G) - N(x) \). Since \( S \) is an independent set of \( n - k \) vertices and \( H' \) is \( r-1 \)-partite, thus \( H \) would be \( r \)-partite.

**Claim 5** \( e(H) \geq e(G) \)

By construction, \( e(H) = e(H') + k(n - k) \). We also have \( e(G) \leq e(G') + \sum_{v \in S} d_G(v) \) as the difference of edges between \( G \) and \( G' \) would only consist of those edges that have at-least one end-point in the set \( S = V(G) - V(G') \). Note that the edges with both end-points in the set \( S \) are counted twice. Since \( \Delta(G) = k \), we have \( d_G(v) \leq k \) for each \( v \in S \). As \( |S| = n - k \), we have \( \sum_{v \in S} d_G(v) \leq k(n - k) \). Therefore, we have

\[
e(G) \leq e(G') + \sum_{v \in S} d_G(v) \leq e(G') + k(n - k) \leq e(H') + k(n - k) = e(H).
\]

**Example:** *Distant pairs of points* [1]

In a circular city of diameter 1, we might want to locate \( n \) police cars to maximize the number of pairs that are far apart, say separated by distance more than \( d = 1/\sqrt{2} \). If six cars occupy equally spaced points on the circle, then the only pairs not more than \( d \) apart are the consecutive pairs around the outside: there are nine good pairs. Instead, putting
two cars each near the vertices of an equilateral triangle with side-length $\sqrt{3}/2$ yields three bad pairs and twelve good pairs. (This may not be the socially best criterion!) In general, with $\lceil n/3 \rceil$ or $\lfloor n/3 \rfloor$ cars near each vertex of this triangle, the good pairs correspond to edges of the tripartite Turan graph.

Figure 4: [1]

References


